Bloch-Floquet band gaps for water waves over a periodic bottom

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CIMAV 2025
Aussois – mai 2024

Outline

1. **Introduction**
   - The Water-Wave problem
   - Linearized system near water at rest
   - Simple facts about the DNO
   - Linear waves over a periodic bottom
2. **Bloch-Floquet transform for DNO**
   - Generality on Bloch-Floquet transform
   - Application to DNO
3. **Perturbation of self-adjoint operators**
   - Flat bottom
   - Gap opening

Outline

Water wave problem with variable bottom

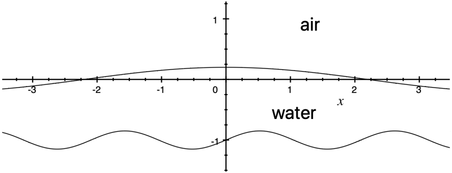
Euler equation for irrotational flow. Free boundary problem.
Time-dependent 2D fluid domain:

$$\Omega(t) = \{(x, y) \in \mathbb{R}^2, -h + b(x) < y < \eta(x, t)\}$$

delimited by a fixed bottom and a free surface $y = \eta(x, t)$.
$u(x, y, t)$: velocity of a fluid particle located at (x, y), at time t.

- Irrotational : $\operatorname{curl} u = 0 \Rightarrow$ Potential flow: $u = \nabla\varphi$
- Incompressible: $\operatorname{div} u = 0 \Rightarrow \quad \Delta\varphi = 0 \text{ in } \quad \Omega(t)$

on $y = \eta(x, t)$:



$$\begin{cases} \partial_t\eta = \partial_y\varphi - \partial_x\eta \cdot \partial_x\varphi \\ \partial_t\varphi = -g\eta - \dfrac{1}{2}|\nabla\varphi|^2 \end{cases}$$

on $y = -h + b(x)$: $\frac{\partial\varphi}{\partial n} = 0$

Transforming to a system for surface variables

$\eta =$ Surface elevation

$\xi = \varphi(x, \eta(x,t), t)$ = Trace of velocity potential on surface

$$\partial_t \eta = G[\eta, b]\xi$$

$$\partial_t \xi = -g\eta - \frac{1}{2}(\partial_x \xi)^2 + \frac{1}{2}\frac{(G[\eta, b]\xi + \partial_x \eta \partial_x \xi)^2}{1 + (\partial_x \eta)^2}$$

g is the acceleration of gravity.

$G[\eta, b]$ is the Dirichlet – Neumann operator: For φ solution of

$$\begin{cases} \Delta\varphi = 0 \quad \text{in} \quad \Omega \\ \varphi_{|_{y=\eta}} = \xi \,, \quad \partial_n \varphi_{|_{y=-h+b}} = 0 \,. \end{cases}$$

$$\xi := \varphi|_{y=\eta} \;\rightarrow\; G[\eta, b]\xi = \frac{\partial \varphi}{\partial n}|_{y=\eta(x)} \sqrt{1 + |\nabla_x \eta|^2}$$

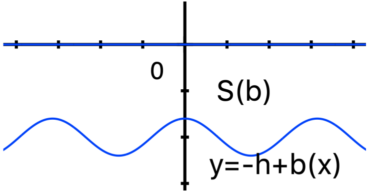
Linearized system near water at rest

2D water wave system linearized near the stationary state $(\eta(x), \xi(x)) = (0, 0)$ over variable bottom $y = -h + b(x)$

$$\begin{cases} \partial_t \eta - G[b]\xi = 0 \\ \partial_t \xi + g\eta = 0, \end{cases}$$

$\eta, \xi : (x, t) \in \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; g : acceleration due to gravity.
G[b] is the Dirichlet-Neumann operator (DNO) for the domain $\Omega(b) = \{(x, y), -h + b(x) < y < 0\}$, $b \in C^2(\mathbb{R})$, bounded.

$$\xi \ \to \ G[b]\xi = \partial_y \varphi_{|_{y=0}} \ , \quad \text{(nonlocal, 1}^{\text{st}} \text{ order operator)}$$



φ solution of Laplace eq.

$$\begin{cases} \Delta\varphi = 0 \quad \text{in} \quad \Omega(b) \\ \varphi_{|_{y=0}} = \xi \ , \quad \partial_n \varphi_{|_{y=-h+b}} = 0 \ . \end{cases}$$

Simple facts about the DNO

$\Omega(b) = \{(x, y), -h + b(x) < y < 0\}$, $\quad b \in C^2(\mathbb{R})$, bounded.

$$\begin{cases} \Delta\varphi = 0 \quad \text{in} \quad \Omega(b) \\ \varphi_{|_{y=0}} = \xi\,, \quad \partial_n \varphi_{|_{y=-h+b}} = 0\,. \end{cases}$$

$$G[b]\xi = \partial_y \varphi_{|_{y=0}}$$

By elliptic regularity, (Lannes 2013)
– $G[b]$ is a continuous operator from $\dot{H}^1(\mathbb{R})$ to $L^2(\mathbb{R})$,
– positive semi-definite,
– symmetric for the L^2-scalar product.
– It is also self-adjoint on $L^2(\mathbb{R})$ with domain $H^1(\mathbb{R})$.

If $b = 0$, (flat bottom)

$$G[0]\varphi = D\tanh(D)\varphi = \lambda\varphi$$

$$\lambda_k = k\tanh(k), \quad \varphi = e^{ikx}, k \in \mathbb{R}$$

The spectrum of $G[0]$ is $[0, \infty)$.

Linear waves over a periodic bottom

2 different situations:

- Finite number of periodic ripples (sandbars): Bragg resonances.

  Bottom $y = h(x)$, with $h(x) = h_1$ if $x \leqslant 0$ and $x \geqslant \ell$
  $h(x)$ periodic for $0 < x < \ell$.

  Reflection of a substantial fraction of the incident wave energy when the wavelength of the incoming wave is twice the wavelength of the ripple.

  - observed in water tank experiments [Heathershaw 1982, Davis-Heathershaw 1984].
  - obtained theoretically [Davies 1982, Mei 1985, Miles 1998, Porter-Porter 2003].
  - explains the formation of multiple long-shore sandbars on gently sloping beaches [Yu-Mei 2000].
  - provides a mechanism of coastal protection by finite array of artificial bars [Mei-Hara-Naciri 1988]. $\rightarrow$ Bar spacing is chosen such that Bragg resonances occur and a large portion of waves at frequencies within resonant bands may be reflected to the sea.

- Periodic bottom of infinite extension

  $\rightarrow$ Spectral problem for the Dirichlet-Neumann operator.

  In analogy with Schrödinger operator with periodic potentials, its spectrum composed of spectral bands separated by gaps, if the bottom variations are sufficiently small.

  - Yu-Howard ('07-'12): Compute numerically the Bloch-Floquet eigenfunctions and eigenvalues for various bottom profiles; identify the spectral gaps.
  - Liu-Liu-Lin (2019): Shallow regime and bottom composed of infinite array of parabolic bars. Relationship between Bragg resonances and Bloch band gap.
  - C.L.-Ménard-Sulem (2024): rigorous construction of the spectrum of DNO on $\mathbb{R}$ as a sequence of bands separated by gaps. Necessary and sufficient conditions on the Fourier coefficients of b(x) for the opening of gaps.

Assumptions on $b(x)$: $(h = 1)$

- $b(x)$ is 2π-periodic, $C^2(\mathbb{T}_{2\pi})$, $\int_0^{2\pi} b(x)dx = 0$.
- $1 - \varepsilon b(x) \geqslant c_0 > 0$. (small variations)

How does a small periodic bottom modify the spectrum of $G[\varepsilon b]$?

Main ingredients of analysis:

– Bloch-Floquet theory: describes the spectrum and the generalized eigenfunctions of $G[\varepsilon b]$ by a family, parametrized by θ, of spectral problems for $G_\theta[\varepsilon b]$ acting of 2π-periodic functions.

– Elliptic estimates, analyticity of $G_\theta[\varepsilon b]$ w.r.t. ε, θ (Lannes 2013)

– Perturbation theory of self-adjoint operators.(Rellich 1969, Reed-Simon 1978, Lewin 2022)

– Notion of quasi-modes: Method to construct eigenvalues from approximate ones (Bambusi-Kappeler-Paul, 2015)

Outline

General Bloch-Floquet transform (1883)

The Bloch-Floquet transform is defined on $\mathcal{S}(\mathbb{R})$ as
(Reed-Simon 1978, Kushment 2016)

$$f(x) \in \mathcal{S}(\mathbb{R}) \mapsto (\mathcal{U}f)(x,\theta) := \sum_{n=-\infty}^{\infty} f(x+2\pi n)e^{-2\pi i\theta n}, \quad \theta \in \mathbb{T}^1.$$

$\mathcal{U}f$ is 'θ-periodic' in x: i.e. $(\mathcal{U}f)(x+2\pi,\theta) = e^{2\pi i\theta}(\mathcal{U}f)(x,\theta)$
for Bloch-Floquet parameter $\theta \in (-\frac{1}{2}, \frac{1}{2}]$.

• $\mathcal{U}$ uniquely extends to a unitary operator from $L^2(\mathbb{R})$ to $L^2((-\frac{1}{2}, \frac{1}{2}]; L^2(0,2\pi))$.

$$\int_{\mathbb{R}} |f(x)|^2 dx = \int_{1/2}^{1/2} \left(\int_0^{2\pi} |(\mathcal{U}f)(x,\theta)|^2 \, \mathrm{d}x \right) \mathrm{d}\theta$$

• For $f \in L^2(\mathbb{R})$, $f(x) = \int_{-1/2}^{1/2} \mathcal{U}f(x,\theta)\,\mathrm{d}\theta$

With our choice of direct integral decomposition of functional spaces

$$L^2(\mathbb{R}) = \int_{(-\frac{1}{2},\frac{1}{2}]}^{\oplus} L^2_\theta \, \mathrm{d}\theta,$$

the goal is to decompose $G[\varepsilon b]$ into operators $G_\theta[\varepsilon b]$ acting of 2π-periodic functions.

Denote $\omega_\varepsilon := \{(x,y),\ x \in \mathbb{T}_{2\pi},\ -1+\varepsilon b(x) < y < 0\}$. (periodic cell)
For $\phi \in H^1(\mathbb{T}_{2\pi})$, let Φ be the unique variational solution of

$$\begin{cases} (-\Delta - 2i\theta\partial_x + \theta^2)\Phi = 0 \quad \text{in } \omega_\varepsilon, \\ \Phi_{|_{z=0}} = \phi, \quad (\partial_n + i\theta n_x)\Phi_{|_{y=-1+\varepsilon b}} = 0, \end{cases}$$

$$G_\theta[\varepsilon b]\phi = \partial_n \Phi_{|_{z=0}} \in L^2(\mathbb{T}_{2\pi})$$

is well defined on $H^1(\mathbb{T}_{2\pi})$, closed, symmetric, positive semi-definite and bounded uniformly with respect to θ.

(a) Important operator: Resolvent operator $(1+G_\theta[\varepsilon b])^{-1}$.
defined through another auxiliary elliptic system on $H^1(\omega_\varepsilon)$.

$(1+G_\theta[\varepsilon b])^{-1}$ bounded from $L^2(\mathbb{T}_{2\pi})$ to $H^1(\mathbb{T}_{2\pi})$, independently of θ, ε.

(b) Consequences.

- $G_\theta[\varepsilon b]$ is self-adjoint with domain $H^1(\mathbb{T}_{2\pi})$
- $\sigma(G_\theta[\varepsilon b]) \subset [0, \infty)$.
- Compact resolvent + self-adjointness $\Rightarrow$ pure discrete spectrum
- $\lambda_n^\varepsilon(\theta)$ real, finite multiplicity.

$$0 \leqslant \lambda_1^\varepsilon(\theta) \leqslant \ldots \leqslant \lambda_n^\varepsilon(\theta) \leqslant \ldots \to \infty$$

- $\sigma((1+G_\theta[\varepsilon b])^{-1}) = \{\tau_n^\varepsilon(\theta)\}_n \in \mathbb{N}, \quad \tau_n^\varepsilon(\theta) = \frac{1}{1+\lambda_n^\varepsilon(\theta)}$

Structure of the spectrum

Theorem (LMS,24)

The spectrum $\sigma(G[\varepsilon b])$ is composed of a union of bands. Namely,

$$\sigma(G[\varepsilon b]) = \bigcup_{p=0}^{\infty} \lambda_p^{\varepsilon}\Big((-\tfrac{1}{2}, \tfrac{1}{2}]\Big),$$

where the $\{\lambda_p^{\varepsilon}(\theta)\}_{p=0}^{\infty}$ are the eigenvalues of $G_\theta[\varepsilon b]$, labeled in increasing order, repeated with their order of multiplicity, and the bands are images of the Lipschitz non-negative functions $\theta \mapsto \lambda_p^{\varepsilon}(\theta)$ on the interval $(-\frac{1}{2}, \frac{1}{2}]$. Moreover, $\lambda_0^{\varepsilon}(0) = 0$.

Comments on Theorem 1.

The spectrum of $G[\varepsilon b]$ is composed of union of bands that *may or may not overlap*.

Outline

The flat bottom case: $b(x) = 0$

The eigenvalues of $G_\theta[0]$ are

$$\kappa_p(\theta) = (p+\theta)\,\mathtt{tanh}(p+\theta),\; p \in \mathbb{Z},$$

and associated eigenfunctions $\phi_p(x) = e^{ipx}$.
$\kappa_p(\theta)$: simple for $-\frac{1}{2} < \theta < 0$ and $0 < \theta < \frac{1}{2}$.
double for $\theta = 0, \frac{1}{2}$

When *reordered by their size*, the eigenvalues are continuous in θ.

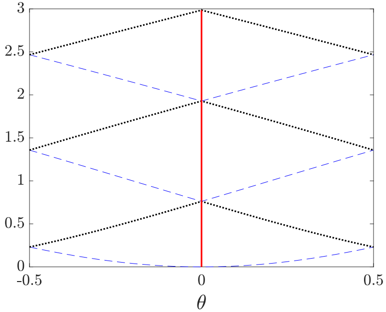
$$\lambda_0^0(\theta) \leqslant \lambda_1^0(\theta) \leqslant \lambda_2^0(\theta) \leqslant \cdots \leqslant \lambda_p^0(\theta) \cdots$$

The eigenvalues of the resolvent $(1 + G_\theta[0])^{-1}$ are

$$\tau_p^0(\theta) = (1 + \lambda_p^0(\theta))^{-1}$$

with the same eigenfunctions.

Sketch of the first six eigenvalues of $G_\theta[0]$



Spectrum of $G[0] = [0, +\infty) = \cup_{n=0}^{+\infty}[\min_\theta \lambda_n^0(\theta), \min_\theta \lambda_n^0(\theta)]$.

Goal: understand how the presence of a small periodic bottom modifies the structure of the spectrum.

Under certain conditions on the Fourier coefficients of b, the presence of the bottom generally results in the splitting of double eigenvalues near points of multiplicity, creating a spectral gap.

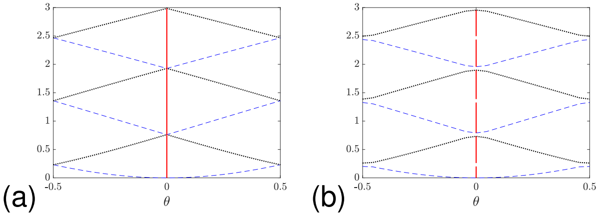


Figure: Sketch of the first six eigenvalues in order of magnitude: (a) flat bottom $\varepsilon = 0$; (b) in the presence of a small generic bottom perturbation $\varepsilon > 0$. The dashed blue (resp. dot black) curve represents $\lambda^{\varepsilon}_{2p}(\theta)$ (resp. $\lambda^{\varepsilon}_{2p+1}(\theta)$).
The spectrum of the operator $G[\varepsilon b]$ is represented by the unions of solid red intervals.

Analyticity of $G_\theta[\varepsilon b]$ and $(1 + G_\theta[\varepsilon b])^{-1}$ w.r.t. ε and θ.
We know $b \mapsto G[b]$ is analytic with respect to b (Lannes, 2013).
Here, we need to keep track of dependency in θ.

Perturbation theory of the spectrum of self-adjoint, analytic operators. (Lewin, 2022).

- For θ not too close to 0 or $\frac{1}{2}$, $p \in \mathbb{N}$, and ε sufficiently small (depending on b and p), $G_\theta[\varepsilon b]$ has a simple eigenvalue $\lambda_p^\varepsilon(\theta)$ in an interval outside the gap we will construct.
- Perturbation of a double eigenvalue: first description of the spectrum of $G_\theta[\varepsilon b]$ for ε small, and θ close to 0 or $\frac{1}{2}$.

Structure of the lower part of the spectrum

Theorem (LMS24)

(i) *For any $p \in \mathbb{N}$, there exists $\varepsilon_1(b,p) \in (0,\varepsilon_0]$ and $C_{b,p}$ such that we have*

$$\mathrm{d}\Big(\lambda_p^{\varepsilon}\Big(\big(-\tfrac{1}{2},\tfrac{1}{2}\big]\Big),\lambda_p^{0}\Big(\big(-\tfrac{1}{2},\tfrac{1}{2}\big]\Big)\Big) \leqslant C_{b,p}\varepsilon, \quad \forall \varepsilon \in [0,\varepsilon_1).$$

(ii) *The lower part of the spectrum of $G[\varepsilon b]$ is purely absolutely continuous. More precisely, for any $M > 0$, there exists $\varepsilon_M \in (0,\varepsilon_0]$ such that for any $\varepsilon \in [0,\varepsilon_M)$,*

$$\sigma(G[\varepsilon b]) \cap [0,M] = \sigma_{\mathrm{ac}}(G[\varepsilon b]) \cap [0,M]$$
$$\sigma_{\mathrm{pp}}(G[\varepsilon b]) \cap [0,M] = \sigma_{\mathrm{sc}}(G[\varepsilon b]) \cap [0,M] = \emptyset.$$

Comments on Theorem 2.

(a) The spectrum of $G[\varepsilon b]$ is composed of union of bands that *may or may not overlap*.

(b) In the case of 1D Schrödinger operators with periodic potentials, bands cannot overlap due to the key property that the eigenvalues, labeled in increasing order are *strictly monotone functions of* θ.
Studying the opening of a gap reduces to studying the splitting of the eigenvalues at $\theta = 0, 1/2$.

(c) For $G_\theta[\varepsilon b]$, the monotonicity of the $\lambda_p(\theta, \varepsilon)$ with respect of θ is unknown.

The next theorems give conditions on the Fourier coefficients of b,

$$\widehat{b}_k = \frac{1}{2\pi}\int_0^{2\pi} b(x)e^{-ikx}\,\mathrm{d}x,$$

that ensure the opening of a gap that separates the double eigenvalues $\lambda^0_{2p-1}(0) = \lambda^0_{2p}(0)$ or $\lambda^0_{2p}(\frac{1}{2}) = \lambda^0_{2p+1}(\frac{1}{2})$ corresponding to flat bottom.

Gap opening at order $O(\varepsilon)$

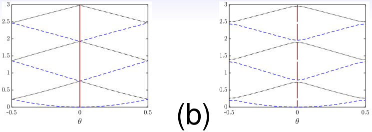


Figure: (a) flat bottom $\varepsilon = 0$; (b) small bottom perturbation.
Blue curve : $\lambda^0_{2p}(\theta), \lambda^\varepsilon_{2p}(\theta)$. Black curve : $\lambda^0_{2p+1}(\theta), \lambda^\varepsilon_{2p+1}(\theta)$.

Theorem Fix $p \in \mathbb{N}$, $p \neq 0$, and assume $\widehat{b}_{2p} \neq 0$. There exist $\varepsilon_2(b,p) > 0$ and $C_{b,p} > 0$ such that for all $\varepsilon \in (0, \varepsilon_2)$, the spectrum $\sigma(G[\varepsilon b])$ has a gap. Denoting :

$$g^-_{2p,\varepsilon} := \lambda^0_{2p}(0) - \max_\theta \lambda^\varepsilon_{2p-1}(\theta), \quad g^+_{2p,\varepsilon} := \min_\theta \lambda^\varepsilon_{2p}(\theta) - \lambda^0_{2p}(0)$$

Then, $\max_\theta \lambda^\varepsilon_{2p-1}(\theta) < \min_\theta \lambda^\varepsilon_{2p}(\theta)$ and

$$\left| g^\pm_{2p,\varepsilon} - F_{2p}|\widehat{b}_{2p}|\varepsilon \right| \leqslant C_{b,p}\varepsilon^2, \qquad F_p := \frac{(p/2)^2}{\cosh^2(p/2)}$$

Gap opening at order $O(\varepsilon)$

Proof in 3 steps:

(a) Construction of approximated eigenvalues of $G_\theta[\varepsilon b]$.

Fix $\theta = \delta\varepsilon$, $\delta \in [0, d_p]$

$$\begin{cases} \lambda^{\mathrm{app}}_{p\pm}(\delta,\varepsilon) := \lambda^0_{2p}(0) + \varepsilon\lambda'_{p\pm}(\delta) \\ U^{\mathrm{app}}_{p\pm}(\delta,\varepsilon,x,z) := U^0_{p\pm}(x,z) + \varepsilon U'_{p\pm}(\delta,x,z). \end{cases}$$

Insert in original elliptic system and identify formally terms of order $O(1), O(\varepsilon)$. Solvability condition:

$$\lambda'_{p\pm}(\delta) = \pm\left(K_p^2\delta^2 + F_{2p}^2|\widehat{b}_{2p}|^2\right)^{\frac{1}{2}}.$$

$$F_{2p} = \left(\frac{p}{\cosh(p)}\right)^2, \quad K_p = \frac{p}{\cosh(p)^2}\left(1 + \frac{\sinh(2p)}{2p}\right).$$

(b) Approximation lemma: Extension of a result for operators on finite dimensional spaces (Bambusi-Kappeler-Paul, 2015), to compact operators on Hilbert spaces. [It will be applied to $(1 + G_\theta[\varepsilon b])^{-1}$].

Lemma

Let K be a compact, positive semi-definite, self-adjoint operator on a separable Hilbert space H. If $(\lambda^{\mathrm{app}}, u^{\mathrm{app}}) \in \mathbb{R}_+ \times H$ satisfies $\|u^{\mathrm{app}}\| = 1$ and $\|Ku^{\mathrm{app}} - \lambda^{\mathrm{app}} u^{\mathrm{app}}\| \leqslant \eta$, there exists an eigenvalue λ of K such that $|\lambda - \lambda^{\mathrm{app}}| \leqslant \eta$.

(c) Preparation to apply lemma with operator $(1 + G_\theta[\varepsilon b])^{-1}$. Denote $\xi^{\mathrm{app}}_{p\pm} := U^{\mathrm{app}}_{p\pm}|_{y=0}$, $\tau^{\mathrm{app}}_{p\pm} := (1 + \lambda^{\mathrm{app}}_{p\pm})^{-1}$. Assuming ε is small enough (depending only on b and p), one has $\forall \delta \in [0, d_{p,1}]$

$$\left\|(1 + G_{\delta\varepsilon}[\varepsilon b])^{-1} \xi^{\mathrm{app}}_{p\pm} - \tau^{\mathrm{app}}_{p\pm} \xi^{\mathrm{app}}_{p\pm}\right\|_{L^2(\mathbb{T}_{2\pi})} \leqslant C_{b,p} \varepsilon^2 \left\|\xi^{\mathrm{app}}_{p\pm}\right\|_{L^2(\mathbb{T}_{2\pi})}.$$

(d) Application of lemma: Let $\delta \in [0, d_{1,p}]$, $\theta = \delta\varepsilon \in [0, d_{1,p}\varepsilon]$. If $\varepsilon < \varepsilon_p$, $\exists$ two e.v. $\tau^{\varepsilon}_{p\pm}(\theta)$ of $(1 + G_\theta[\varepsilon b])^{-1}$ such that

$$|\tau^{\varepsilon}_{p\pm}(\theta) - \tau^{\mathrm{app}}_{p\pm}(\delta, \varepsilon)| \leqslant C_p\varepsilon^2.$$

Consequently, $\exists$ two e.v. $\lambda^{\varepsilon}_{p\pm}(\theta)$ of $G_\theta[\varepsilon b]$ such that

$$|\lambda^{\varepsilon}_{p\pm}(\theta) - \lambda^{\mathrm{app}}_{p\pm}(\delta, \varepsilon)| \leqslant C_p\varepsilon^2.$$

The spectrum of $G_\theta[\varepsilon b]$ has exactly two eigenvalues in a neighborhood of $\lambda^0_{2p}(0)$, therefore $\lambda^{\varepsilon}_{2p-1}(\theta) = \lambda^{\varepsilon}_{p-}(\theta)$ and $\lambda^{\varepsilon}_{2p}(\theta) = \lambda^{\varepsilon}_{p+}(\theta)$.

$$\lambda^{\varepsilon}_{2p}(\theta) \geqslant \lambda^0_{2p}(0) + F_{2p}|\widehat{b}_{2p}|\varepsilon - C_p\varepsilon^2,$$

and similarly for $\lambda^{\varepsilon}_{2p-1}(\theta)$.
Lower bound for the separation between the two eigenvalues of $\sigma(G_\theta[\varepsilon b])$ in the vicinity of $\theta = 0$, if $\widehat{b}_{2p} \neq 0$:

$$\lambda^{\varepsilon}_{2p}(\theta) - \lambda^{\varepsilon}_{2p-1}(\theta) \geqslant 2F_{2p}|\widehat{b}_{2p}|\varepsilon - C_p\varepsilon^2.$$

(e) A last step is needed to show that the gap remains open for θ far from 0. Combine with estimates for perturbations of simple e.v.

(f) If $\widehat{b}_{2p} = 0$, higher order expansion gives opening of gaps of order $O(\varepsilon^2)$ under conditions on Fourier coefficients.

Open Questions

1. We do not know whether $\lambda_p(\theta, \varepsilon)$ are monotone with respect to θ.
2. We do not know whether the number of spectral gaps is infinite. The number of gaps increases as the size of the bottom variations ε gets smaller and smaller.
3. We only proved that the lower part of the spectrum of $G[\varepsilon b]$ is purely absolutely continuous.

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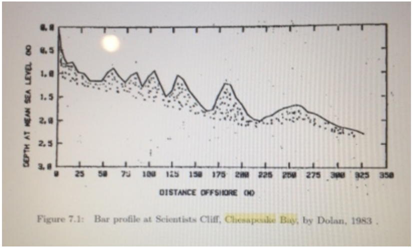
Thank you



Aerial view of submarine longshore sandbars, Escambia Bay, Florida,

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