Approximation numérique des ondes en géophysique

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CIMAV, 13 mai 2025

Primitive equations employed in ocean modelling (3-D)

Momentum:

$$
\begin{cases}
\dfrac{Du}{Dt} + fw \underbrace{- fv + \dfrac{1}{\rho}\dfrac{\partial P}{\partial x}}_{\text{Geostrophic equilibrium}} = F_x \\
\dfrac{Dv}{Dt} + \underbrace{fu + \dfrac{1}{\rho}\dfrac{\partial P}{\partial y}}_{\text{Geostrophic equilibrium}} = F_y \\
\dfrac{Dw}{Dt} - fu + \underbrace{\dfrac{1}{\rho}\dfrac{\partial P}{\partial z} + g}_{\text{Hydrostatic equilibrium}} = F_z
\end{cases}
$$

Continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

Tracers:

$$
\begin{cases}
\rho C_v \dfrac{DT}{Dt} + p\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}\right) = k_T \nabla^2 T \\
\dfrac{DS}{Dt} = k_S \nabla^2 S
\end{cases}
$$

State:

$$\rho - \rho_0\left(1 - \alpha(T - T_0) + \beta(S - S_0)\right) = 0$$

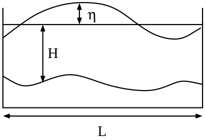
and appropriate initial and boundary conditions. Velocity (u, v, w), Pressure P, Temperature T, Salinity S, Density ρ.

The inviscid non linear shallow-water equations (non conservative form)

They are derived by vertical integration of the momentum and continuity equations in the primitive system assuming:

- Horizontal displacements i.e. $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$
- $H \ll L$
- The density ρ is constant
- The hydrostatic equilibrium ($\frac{\partial p}{\partial z} = -\rho g$)
- H remains constant in the following

$$
\begin{aligned}
\frac{D\mathbf{u}}{Dt} + f\mathbf{k} \times \mathbf{u} + g\nabla\eta &= 0, \\
\frac{D\ln(H+\eta)}{Dt} + \nabla \cdot \mathbf{u} &= 0.
\end{aligned}
$$



Inertia-gravity and Rossby waves in 2–D

We consider the linear SW system

$$\begin{aligned} \mathbf{u}_t + f\mathbf{k}\times\mathbf{u} + g\nabla\eta &= 0, \\ \eta_t + H\nabla\cdot\mathbf{u} &= 0, \end{aligned}$$

Periodic solutions

$$\begin{aligned} \mathbf{u} &= \hat{\mathbf{u}}e^{i(kx+ly+\omega t)}, \\ \eta &= \hat{\eta}e^{i(kx+ly+\omega t)}. \end{aligned}$$

After substitution, the dispersion relation, i.e. $\omega(k,l)$, is obtained

- $\omega = 0$: The (slow) geostrophic mode (for f constant).
- $\omega = \pm\sqrt{f^2 + gH(k^2+l^2)}$: The (fast) inertia-gravity modes

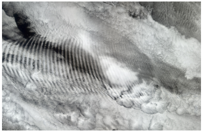
  Two limits: • Pure gravity waves : $\omega = \pm\sqrt{gH(k^2+l^2)}$ when $f=0$.

  • Pure inertial oscillations $\omega = \pm f$ when $gH(k^2+l^2) \ll f^2$.

By using the quasi-geostrophic approximation we can also obtain a relation for the

- Rossby mode: $\omega = \dfrac{-\beta k}{\frac{1}{\lambda^2} + k^2 + l^2}$, with $f = f_0 + \beta y$, $f_0 = 2\Omega\sin\varphi_0$ and $\lambda = \sqrt{gH}/f_0$.

High frequency inertia-Gravity



Low frequency Rossby waves

Let $\mathbf{u}_h = (u_h, v_h)$ and η_h belong to appropriate spaces $\mathbf{V}_h$ and W_h, resp., with test functions $\boldsymbol{\phi}_h$ and ψ_h. Let $\{\tau_h\}_{h>0}$ denote a partition of the domain Ω into a finite number N of elements K_{el}.

Continuous variational formulation

$$\sum_{el=1}^{N} \int_{K_{el}} \frac{\partial \mathbf{u}_h}{\partial t} \cdot \boldsymbol{\phi}_h \, d\mathbf{x} + \sum_{el=1}^{N} \int_{K_{el}} f\mathbf{k} \times \mathbf{u}_h \cdot \boldsymbol{\phi}_h \, d\mathbf{x} + \sum_{el=1}^{N} \int_{K_{el}} g \nabla \eta_h \cdot \boldsymbol{\phi}_h \, d\mathbf{x} = 0,$$

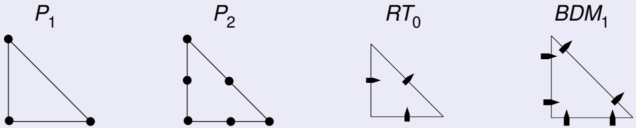
$$\sum_{el=1}^{N} \int_{K_{el}} \frac{\partial \eta_h}{\partial t} \psi_h \, d\mathbf{x} + \sum_{el=1}^{N} \int_{K_{el}} H \nabla \cdot \mathbf{u}_h \psi_h \, d\mathbf{x} = 0.$$

$\forall \boldsymbol{\phi}_h \in \mathbf{V}_h, \forall \psi_h \in W_h$. Appropriate initial and boundary conditions are taken into account.

General framework: consider continuous linear forms

$$\begin{cases} a(\mathbf{u}_h, \boldsymbol{\phi}_h) + b(\boldsymbol{\phi}_h, \eta_h) = < \mathbf{F}, \boldsymbol{\phi}_h >_{\mathbf{V}' \times \mathbf{V}} & \forall \boldsymbol{\phi}_h \in \mathbf{V}_h, \\ b(\mathbf{u}_h, \psi_h) + d(\eta_h, \psi_h) = < G, \psi_h >_{W' \times W} & \forall \psi_h \in W_h. \end{cases} \qquad \begin{pmatrix} A_h & -B_h^t \\ B_h & D_h \end{pmatrix} \begin{pmatrix} \mathbf{u}_h \\ \eta_h \end{pmatrix} = RHS.$$

P and RT and BDM finite elements

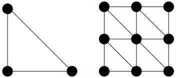


Fourier analysis at the discrete level

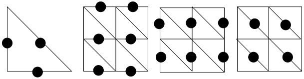
- Time is assumed to be continuous ($\frac{\partial}{\partial t} = i\omega$) and f is held constant.
- The discrete problem leads to a set of discrete equations in space (at node $j_1 = 1, 2, 3, \cdots$, for $\mathbf{u}$ and $j_2 = 1, 2, 3, \cdots$, for η) on a regular and uniform mesh (the meshlength parameter h is taken as a constant). In the following biased right triangles are used.
- Periodic solutions $\mathbf{u}_{j_1} = \hat{\mathbf{u}}_p \, e^{i(kx_{j_1} + ly_{j_1} + \omega t)}$ and $\eta_{j_2} = \hat{\eta}_q \, e^{i(kx_{j_2} + ly_{j_2} + \omega t)}$ are sought where $\hat{\mathbf{u}}_p$ and $\hat{\eta}_q$ are the Fourier amplitudes, with $p = 1, 2, 3 \cdots$, and $q = 1, 2, 3, \cdots$.
- When linear polynomials are employed to approximate $\mathbf{u}$ and η, the velocity and pressure unknowns are located at triangle vertices and we have $p = q = 1$ (for symmetry reasons).

  However, when mid-side, barycenter, internal, etc ..., nodes are used to locate velocity and surface-elevation nodal values we have $p > 1$ and $q > 1$. For example:

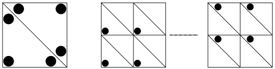
P_1 $(p, q = 1)$



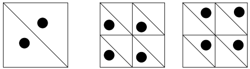
RT_0, P_1^{NC} $(p, q = 3)$



P_1^{DG} $(p, q = 6)$

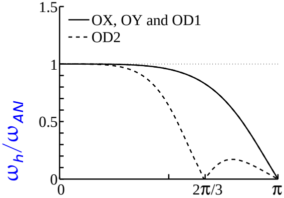


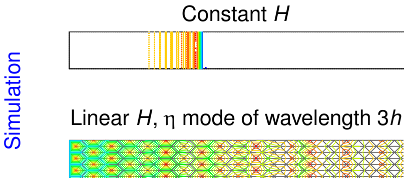
P_0 $(p, q = 2)$



A $n \times n$ system is obtained for the amplitudes and the dispersion relation is hence a polynomial of degree $n = 2p + q$ or $n = p + q$ in ω, leading to the existence of eventual spurious solutions.

$P_m - P_m$ schemes: spurious elevation modes ($\dim\ker(B_h^t) \neq 1$), e.g. $m = 1$





- The behavior of the smallest nonzero singular value σ_0 of the discrete divergence operator is related to the so-called discrete stability inf-sup (LBB) condition for mixed problems

$$\inf_{\psi_h \in W_h} \sup_{\boldsymbol{\phi}_h \in \mathbf{V}_h} \frac{b(\boldsymbol{\phi}_h, \psi_h)}{\|\boldsymbol{\phi}_h\|_{\mathbf{V}} \;\; \|\psi_h\|_{W/\ker B^t}} \geq \sigma_0 > 0,$$

where B is the linear continuous operator defined as
$< B\mathbf{u}, \psi >_{W' \times W} = b(\mathbf{u}, \psi) = \int_\Omega \nabla \cdot \mathbf{u}\, \psi\, d\mathbf{x}, \;\; \forall \mathbf{u} \in \mathbf{V}, \forall \psi \in W.$

- $\sigma_0 \neq 0$ is needed when $\dim(\mathbf{V}_h)$ and $\dim(W_h)$ increase, to avoid a zero eigenvalue of the problem associated with a stationary spurious η mode ($\mathbf{u} = \mathbf{0}$, $\eta \in \ker(B_h^t)$, $\eta \neq$ **constant**).
- Stabilized FEM (Hughes et al., 1986): retrieving the information lost by the projection Π_{V_h}, i.e. $\mathbf{grad}\,\eta_h - \Pi_{V_h}\mathbf{grad}\,\eta_h$, for a bad choice of V_h and W_h (when there are not enough $\mathbf{u}_h$ compared to η_h), as it is the case when $\mathbf{grad}$ is not injective, namely $\dim(\ker B_h^t) > 1$.

$P_m - P_n$ schemes with $m > n$: spurious inertial oscillations: $\omega = \pm f$

Theorem (DLR, J.Comp.Phys. 2012) : For all FE pairs with $n = 2p + q$ and $p \geq q$, the general dispersion relation is a polynomial of degree n, such that

$$\omega^q \left(\omega^2 - f^2\right)^{p-q} \mathrm{P}_{2q}(\omega) = 0,$$

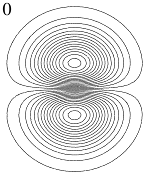
where $\mathrm{P}_{2q}(\omega)$ is a polynomial of degree $2q$ in ω (inertia-gravity solutions).

Consequences: Such FE pairs are subject to

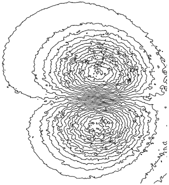
- Physical geostrophic modes $\omega = 0$ of multiplicity q.
- Non physical solutions $\omega = \pm f$, namely spurious inertial modes of multiplicity $p - q$.

Rossby mode $f = f_0 + \beta y$, u (9 periods)

At $t = 0$

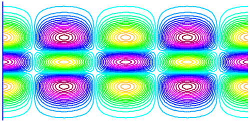


$P_2 - P_1$

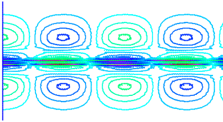


Rossby wave of index 2, $f = \beta y$, v (5 periods)

Continuous: -1.14, 1.14



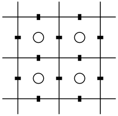
$P_2 - P_1$: -4.45, 4.45



on a structured mesh

Potential problems with the choice $\mathbf{u}_h \in H(div)$

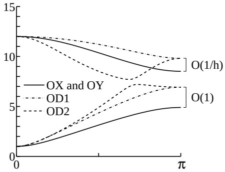
- Coriolis $f-$modes: C-grid and RT, BDM and $BDFM$ elements: $\dim \ker(C) \neq 0$

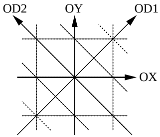


		Inertia part		Gravity part
$\left(\frac{\omega}{f}\right)^2_{AN}$	=	1	+	$\left(\frac{\lambda}{h}\right)^2 \left([kh]^2 + [lh]^2\right)$
$\left(\frac{\omega}{f}\right)^2_{C}$	=	$\cos^2 \frac{kh}{2} \cos^2 \frac{lh}{2}$	+	$4 \left(\frac{\lambda}{h}\right)^2 \left(\sin^2 \frac{kh}{2} + \sin^2 \frac{lh}{2}\right)$

- The choice of the space for η needs be compatible in order to avoid spurious pressure modes. For example, the $RT_0 - P_1$ and $BDM_1 - P_1$ schemes have spurious η modes.
- The $RT_n - P_n^{DG}$ elements on triangles have gaps and spurious inertia-gravity branches

$RT_0 - P_0$: $\frac{\omega_h}{f}$ fine mesh





DLR et al., SIAM J. Sci. Comput., 2007.

- The $RT_n - Q_n^{DG}$ elements on quads have gaps for $n \geq 2$ but no spurious inertia-gravity branches.

- The $BDM_n - P_n^{DG}$ pairs on triangles have gaps and spurious Rossby branches.
- The $BDFM_1 - P_1^{DG}$ element on triangles has gaps but no spurious branches.
- $\mathbf{u}_h \in H(div)$: existence of the discrete Helmholtz decomposition requires that the following diagram commutes, where S is a *streamfunction* space $\subset H^1$, $V \subset H(div)$ and $W \subset L^2$

$H^1(\Omega)$	$\xrightarrow{\nabla^\perp}$	$H(div,\Omega)$	$\xrightarrow{\nabla\cdot}$	$L^2(\Omega)$	deRham complex (compatible Galerkin, mixed FE)
$\downarrow \pi_S$		$\downarrow \pi_V$		$\downarrow \pi_W$	D. Arnold et al. Acta Numerica, 2006
S	$\xrightarrow{\nabla^\perp}$	V	$\xrightarrow{\nabla\cdot}$	W	C. Cotter and E. Shipton, J.Comp.Phys., 2012

where $\nabla^\perp = (-\partial_y, \partial_x)$.

For example, to avoid spurious pressure (η) modes it is required that $\pi_W(\nabla \cdot \mathbf{u}) = \nabla \cdot \pi_V(\mathbf{u})$.

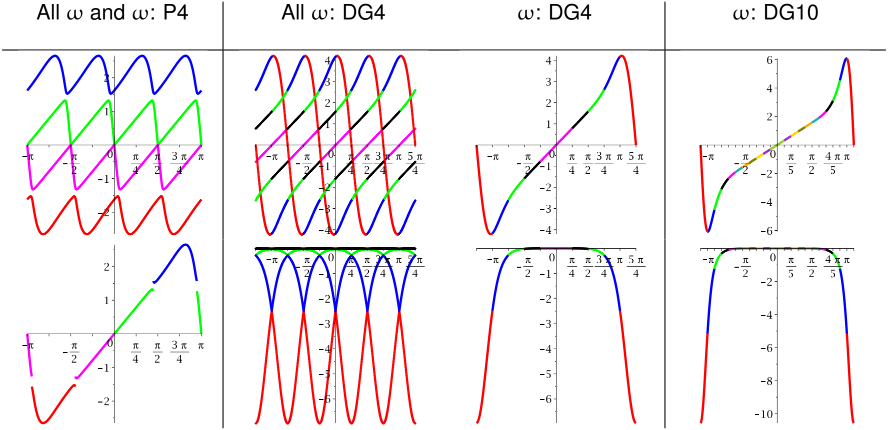
It is found that only the $BDFM_1 - P_1^{DG}$ element satisfies the required embedding properties and avoid spurious branches. However it generates spectral gaps.

Such a commutative diagram does not exist for $P_m - P_n$ schemes.

- Other choices are possible: bubles (dissipation ?), $\mathbf{u}_h \in H(curl)$: D-grid type.

Existence of spectral gaps (DLR et al., SIAM J. Numer. Anal., 2020)

- Solve the advection equation with constant coefficient.
- Use a Fourier analysis for continuous and discontinuous Galerkin approaches and employ polynomials of degree n.
- Using upwinding for the discontinuous method.
- Plot the normalized frequency. The slope at the end of the spectrum is $-(2n+1)$.



The P_1^{DG} and P_1^{NC} schemes

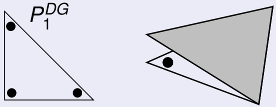
Discontinuous variational formulation (disjoint open triangular elements K_{el})

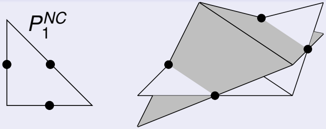
$$\sum_{el=1}^{N}\int_{K_{el}}\frac{\partial \mathbf{u}_h}{\partial t}\cdot\boldsymbol{\phi}_h\,d\mathbf{x}+\sum_{el=1}^{N}\int_{K_{el}} f\mathbf{k}\times\mathbf{u}_h\cdot\boldsymbol{\phi}_h\,d\mathbf{x}-\sum_{el=1}^{N}\int_{K_{el}} g\eta_h\nabla\cdot\boldsymbol{\phi}_h\,d\mathbf{x}+\sum_{el=1}^{N}\int_{\partial K_{el}} g\eta^*\mathbf{n}\cdot\boldsymbol{\phi}_h\,ds=0,$$

$$\sum_{el=1}^{N}\int_{K_{el}}\frac{\partial \eta_h}{\partial t}\phi_h\,d\mathbf{x}-\sum_{el=1}^{N}\int_{K_{el}} H\mathbf{u}_h\cdot\nabla\phi_h\,ds+\sum_{el=1}^{N}\int_{\partial K_{el}} H\mathbf{u}^*\cdot\mathbf{n}\,\phi_h\,ds=0.$$

$\forall\boldsymbol{\phi}_h\in\mathbf{V}_h$, and η^* and $\mathbf{u}^*$ denote the trace of η and $\mathbf{u}$ on ∂K_{el} (stability and consistency).

Discontinuous elements





$$\begin{pmatrix} g\eta^* n_{ed}^x \\ g\eta^* n_{ed}^y \\ H\mathbf{u}^*\cdot\mathbf{n}_{ed}\end{pmatrix}=\begin{pmatrix} g\,n_{ed}^x\{\eta\} & -\frac{\sqrt{gH}}{2}\left(\gamma\,[u]+(1-\gamma)\,n_{ed}^x\,[\mathbf{u}]\cdot\mathbf{n}_{ed}\right) \\ g\,n_{ed}^y\{\eta\} & -\frac{\sqrt{gH}}{2}\left(\gamma\,[v]+(1-\gamma)\,n_{ed}^y\,[\mathbf{u}]\cdot\mathbf{n}_{ed}\right) \\ H\{\mathbf{u}\}\cdot\mathbf{n}_{ed} & -\frac{\sqrt{gH}}{2}[\eta] \end{pmatrix}$$

- $\{\cdot\}$ and $[\cdot]$: mean and jump
- Rusanov: $\gamma=1$
- Roe: $\gamma=0$
- Force: $\gamma=1/2$.

PVM method : M. Castro-Díaz and E. Fernández-Nieto, SIAM J. Sci. Comput., 2012.

To obtain the 2D dispersion relations:

- Perform the 2D Fourier analysis at the discrete level and derive the dispersion relations.
- From equations of degree 18 and 9 in $\omega(kh, lh)$, obtain the asymptotics as $h \longrightarrow 0$.

Theorem (DLR, J.Comp.Phys. 2024):

In the limit as mesh spacing $h \to 0$ we obtain the asymptotic results

• The **inertia-gravity modes** (for all γ): No spurious pressure and no f-modes

$$\omega^{DG} = \omega^{AN} + i\,\mathscr{F}_1(k,l)\,h^3 \pm \mathscr{F}_2(k,l)\,h^4 + O(h^5),$$

$$\omega^{NC} = \omega^{AN} \pm \mathscr{F}_3(k,l)\,h^4 + i\,\mathscr{F}_4(k,l)\,h^5 + O(h^6)$$

• The **geostrophic mode**: No spurious geostrophic modes, except for the Roe scheme

for $\gamma \neq 0$

$$\omega^{DG} = i\,\mathscr{F}_5(k,l)\,h^3 + O(h^5)$$

$$\omega^{NC} = i\,\mathscr{F}_7(k,l)\,h^5 + O(h^7)$$

Roe: $\gamma = 0$

$$\omega^{DG} = 0 \quad \text{and} \quad i\,\mathscr{F}_6(k,l)\,h + O(h^2)$$

$$\omega^{NC} = i\,\mathscr{F}_8(k,l)\,h + O(h^2)$$

where $\mathscr{F}_j$, $j = 1, 2, 3, \ldots$, are polynomial functions which only depend on k, l, and γ.

These results were also obtained very faithfully using numerical simulations (with FENICS).

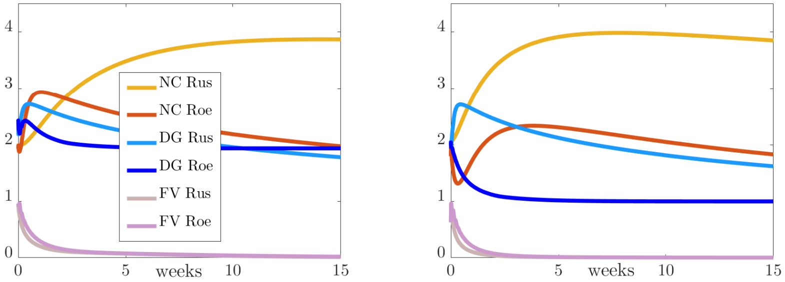
- The test examines the evolution of the evolution of an oceanic eddy at midlatitudes.
- The parameter f is held constant, and the solution should preserve the steady state.

At selected time steps n, a convergence analysis is performed by computing the ratio

$$R_\sigma(t^n = n\Delta t) = \frac{\| \sigma(t^0) - \sigma_h(t^n) \|_{L^2}}{\| \sigma(t^0) - \sigma_{h/2}(t^n) \|_{L^2}}, \quad n = 1, 2, 3, ...,$$

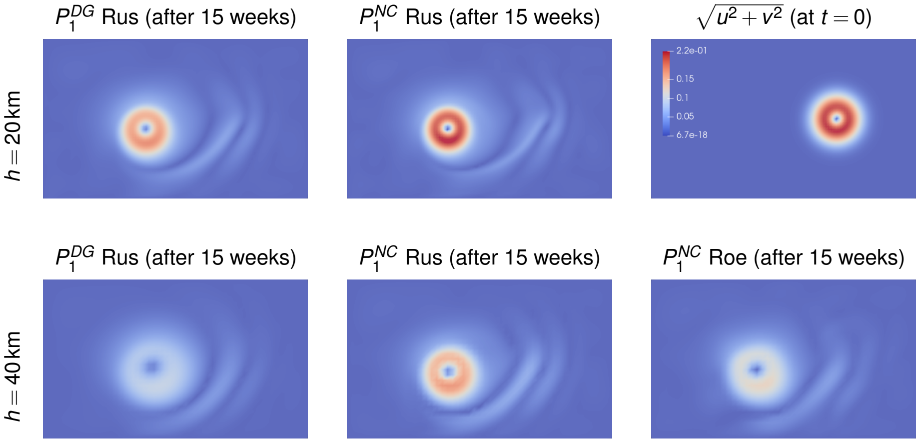
where σ is equal to either η or the flow-speed field $\sqrt{u^2+v^2}$.

$\sigma = \eta$ $\qquad\qquad$ $\sigma = \sqrt{u^2+v^2}$



$\ln \mathscr{R}_\sigma / \ln 2$ with $h/2 = 20\,\mathrm{km}$ up to 15 weeks of simulation on a uniform mesh. A RK4 temporal scheme is used as for the next non linear simulations.

Simulations: non-linear SW equations (conservative form) with $f = f_0 + \beta y$



The flow-speed field $\sqrt{u^2+v^2}$ at $t = 0$ and after 15 weeks of propagation. The legend at $t = 0$ is kept unchanged up to 15 weeks of simulation.

Conclusions

- The discretization of the shallow-water equations usually leads to computational modes.
- We have proposed to study these problems by using Fourier (dispersion) analyses.
- The cause of the computational solutions is mainly due to:
  - Wrong choice of discrete spaces for the variables u, v and η (spurious η modes).
  - An imbalance between the d.o.f. of u, v and η nodal values (inertial modes).
  - The use of normal velocities in $H(div)$ (f-modes, spectral gaps).
- The Fourier analyses show that stabilized DG methods are free of spurious solutions:
  - The P_1^{NC} approximation with $\gamma \neq 0$ is highly accurate for all modes.
  - Finally, we have obtained numerically a CFL limit of:
    - ★ 0.18 for the P_1^{DG} scheme
    - ★ 0.30 for the P_1^{NC} scheme.

    for both the shallow water and advection-diffusion equations.
- Fourier analysis should be performed for 3D models.