Derivation, analysis and numerical simulation of a new high-order shallow-water wave model with canonical non-local Hamiltonian structure in uniform water depth

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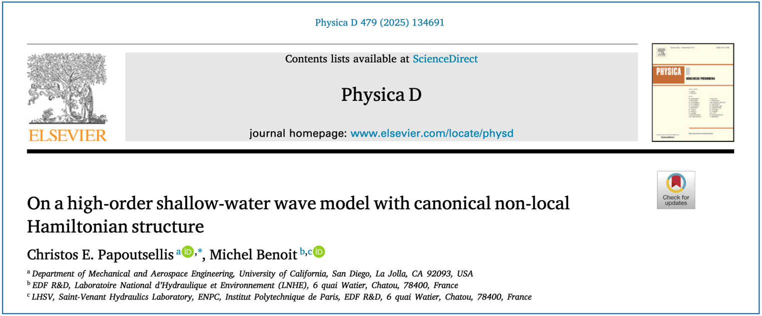
Outline

- Introduction and some remarks
- Variational Modeling
- Comparison with asymptotic models
- Numerical examples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On a high-order shallow-water wave model with canonical non-local Hamiltonian structure

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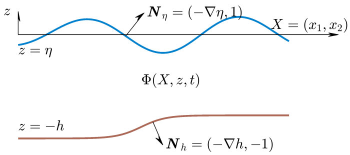
"Now, the next waves of interest, that are easily seen by everyone and which are usually used as an example of waves in elementary courses, are water waves. As we shall soon see, they are the worst possible example, because they are in no respects like sound and light; they have all the complications that waves can have."

—Richard Feynman, *The Feynman Lectures on Physics*, Volume I, Chapter 51

Physical Assumptions and Mathematical Formulations

- The fluid is *ideal* and *homogeneous*
- The fluid is *incompressible*
- The flow is *irrotational*
- The free surface is impermeable
- The bottom surface is impermeable



Physical Assumptions and Mathematical Formulations

- $\partial_t \Phi + \frac{1}{2}(\nabla_{X,z}\Phi)^2 + gz = 0$, on $z = \eta(X,t)$
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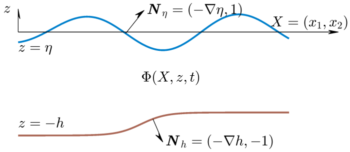

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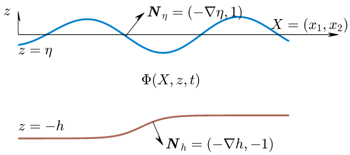

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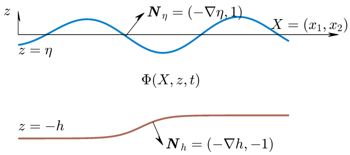

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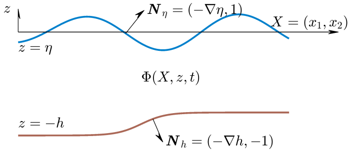
- The bottom surface is impermeable



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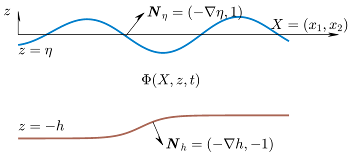

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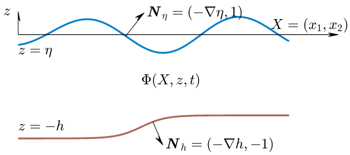

Luke's Variational Principle (unconstrained pressure-type functional)

$$\delta \mathcal{S} = 0, \text{ with } \mathcal{S}[\eta,\Phi] = \int\int\int_{-h}^{\eta}\left[\partial_t\Phi + \frac{1}{2}(\nabla_{X,z}\Phi)^2 + gz\right]dz\,dX\,dt.$$

Luke 1967, A variational principle for a fluid with a free surface, J. Fluid Mech.

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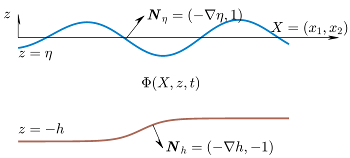

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HAMILTON's equations (non-local) on $(\eta(X,t), \psi(X,t) \overset{\text{def}}{=} \Phi(X,\eta,t))$

$$\partial_t\eta = \mathcal{G}[\eta,h]\psi \overset{\text{def}}{=} \boldsymbol{N}_\eta\cdot[\nabla_{X,z}\Phi]_\eta$$

$$\partial_t\psi = -\frac{1}{2}|\nabla\psi|^2 + \frac{\left(\mathcal{G}[\eta,h]\psi + \nabla\psi\cdot\nabla\eta\right)^2}{2\,(1+|\nabla\eta|^2)} - g\eta$$

$$(\text{DtN})\begin{cases} \Delta_{X,z}\Phi = 0 \\ \boldsymbol{N}_h\cdot[\nabla_{X,z}\Phi]_{-h} = 0 \\ [\Phi]_\eta = \psi \end{cases}$$

LUKE 1967, A variational principle for a fluid with a free surface, J. Fluid Mech.
ZAKHAROV 1968, Stability of periodic waves of finite amplitude on the surface of a deep fluid, J. Appl. Mech. Tech. Phys.
CRAIG & SULEM 1993, Numerical simulation of gravity waves, J. Comp. Phys.

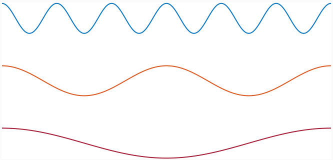
Some basic properties

Dispersion (Cauchy 1815, Airy 1841)

$$\eta = A\cos(kx - \omega t)$$

$$\Phi = \frac{Ag}{\omega}\frac{\cosh(k(z+h))}{\cosh(kh)}\sin(kx - \omega t)$$

$$\boxed{\omega^2(k) = gk\tanh(hk)}$$

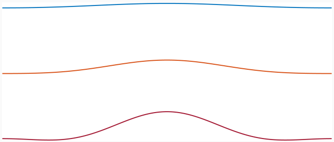


Nonlinearity (Stokes 1847)

$$\eta = \alpha\cos(kx - \omega t) + \mu_2\alpha^2\cos(2(kx - \omega t))$$

$$\Phi = \nu_1\alpha\cosh(k(z+h))\sin(kx - \omega t) + \nu_2\alpha^2\cosh(2k(z+h))\sin(2(kx - \omega t))$$

$$\boxed{\omega^2_{\text{Stokes}}(k\alpha) = \omega^2\left(1 + f(kh)(k\alpha)^2\right)}$$

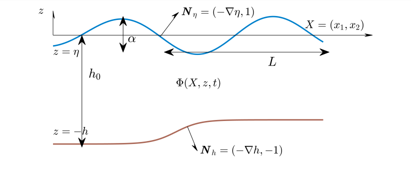


Three typical lengths:

Characteristic depth h_0, horizontal scale L, free-surface amplitude α

Two important non-dimensional parameters:

nonlinearity $\varepsilon = \frac{\alpha}{h_0}$, **shallowness** $\mu = \left(\frac{h_0}{L}\right)^2$



$$\partial_t \eta + \nabla \cdot \left(H\overline{V}\right) = 0$$

$$\partial_t \overline{V}^\star + \nabla\left[\eta + \frac{1}{2}\overline{V}^2 - \mu\left(\frac{1}{3H}\overline{V}\cdot\nabla(H^3\nabla\cdot\overline{V}) + \frac{1}{2}H^2(\nabla\cdot\overline{V})^2\right)\right] = 0$$

$$\overline{V}^\star = \overline{V} - \mu\frac{1}{3H}\nabla(H^3\nabla\cdot\overline{V})$$

- Derivation: asymptotic expansion and discarding $O(\mu^2)$-terms ⇨ correct to $O(\mu)$
- Three evolution Eqs. & one elliptic PDE
- Good dispersive properties but needs improvements in some cases
- Extensions to higher-orders in μ are tricky and involve numerically challenging high-order derivatives

Serre 1953 Contribution à l'étude des écoulements permanents et variables dans les canaux, La Houille blanche
Su & Gardner 1969 KdV eq. and generalizations. Part III. Derivation of KdV and Burgers equation, J. Math. Phys.
Green & Naghdi 1976 A derivation of equations for wave propagation in water of variable depth, J. Fluid Mech.

Variational Modeling approach

$$\mathcal{S}[\eta,\Phi] = \int\int\int_{-h}^{\eta}\left[\partial_t\Phi + \frac{1}{2}\left(\nabla_{X,z}\Phi\right)^2 + gz\right]dz\,dX\,dt.$$

Ansatz: $\Phi^a(X,z,t) \stackrel{\text{def}}{=} \sum_{n=0}^{K}\phi_n(X,t)\,Z_n(z;\eta(X,t),h(X,t)) \equiv \phi_n Z_n$

Then, the $(K+2)$ Euler-Lagrange equations on (η,ϕ_n) are

$$(m=0,\ldots,K)\qquad \left(\partial_t\eta - \boldsymbol{N}_\eta\cdot\left[\nabla_{X,z}\Phi^a\right]_\eta\right)\left[Z_m\right]_\eta + L_{mn}\,\phi_n = 0,$$

$$\left[\partial_t\Phi^a\right]_\eta + g\eta + \frac{1}{2}\left[\nabla_{X,z}\Phi^a\right]_\eta^2 - \left(\partial_t\eta - \boldsymbol{N}_\eta\cdot\left[\nabla_{X,z}\Phi^a\right]_\eta\right)\left[\partial_\eta\Phi^a\right]_\eta - \left(\mathit{\Lambda}_{mn}\,\phi_n\right)\phi_m = 0.$$

$$\begin{aligned} L_{mn} =& A_{mn}\Delta + \underline{B}_{mn}\cdot\nabla + C_{mn},\\ A_{mn} &= (Z_n, Z_m) = A_{nm},\\ \underline{B}_{mn} &= 2(\nabla Z_n, Z_m) + \nabla h\left[Z_n Z_m\right]_{-h},\\ C_{mn} &= (\Delta_{X,z}Z_n, Z_m)\\ &\quad - \boldsymbol{N}_h\cdot\left[\nabla_{X,z}Z_n Z_m\right]_{-h}. \end{aligned}$$

$$\begin{aligned} \mathit{\Lambda}_{mn} =& \alpha_{mn}\Delta + \underline{\beta}_{mn}\cdot\nabla + \gamma_{mn},\\ a_{mn} &= (Z_n, \partial_\eta Z_m),\\ \underline{\beta}_{mn} &= 2(\nabla Z_n, \partial_\eta Z_m) + \nabla h\left[Z_n\partial_\eta Z_m\right]_{-h},\\ c_{mn} &= (\Delta_{X,z}Z_n, \partial_\eta Z_m)\\ &\quad - \boldsymbol{N}_h\cdot\left[\nabla_{X,z}Z_n\partial_\eta Z_m\right]_{-h}. \end{aligned}$$

Advantage: The model will involve spatial derivatives up to second order only.
Caveat: We need to choose "good" vertical functions $Z_n(z;\eta,h)$.

Variational Modeling approach

How to choose Z_n in $\Phi^a(X,z,t) = \sum_{n=0}^{K} \phi_n(X,t)\, Z_n(z;\eta(X,t),h(X))$?

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- ☞ From Boussinesq-Rayleigh long-wave asymptotic expansion:
  ISOBE & KAKINUMA 1994, 2001, IGUCHI et al. 2018

$$\Phi(X,z,t) = \sum_{n=0}^{K} (-1)^n \mu^n \frac{1}{(2n)!} \Delta^n f(X,t)\, (1+z)^{2n} + O(\mu^{K+1})$$

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- ☞ Constant function & parabolic or cosh:
  Klopman et al. 2010, etc.

$$Z_0 = 1, \quad Z_1 = \frac{1}{2}(z-\eta)\frac{2h+z+\eta}{h+\eta}$$

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- ☛ From the linear solution & boundary functions (modes)
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- ☛ Transformation to flat strip & Chebyshev polynomials
  Tian & Sato 2008, Yates, Benoit et al. 2015,...

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Here, we pick functions Z_n from the long-wave asymptotic solution of the DtN problem

The shallow-water ansatz in uniform water depth

The DtN problem in the ZAKHAROV-CRAIG-SULEM formulation with h constant:

$$\mu\Delta\Phi + \partial_z^2\Phi = 0, \quad [\partial_z\Phi]_{-1} = 0, \quad [\Phi]_{\varepsilon\eta} = \psi,$$

can be solved asymptotically in terms of the *shallowness parameter* $\mu = \left(\frac{h_0}{L}\right)^2$:

$$\Phi = \psi + \mu\left\{\Delta\psi + \mu\left[\frac{1}{2}\Delta(H^2) + \nabla(H^2)\cdot\nabla + \frac{H^2}{3}\Delta\right]\Delta\psi\right\}\tilde{Z} + \mu^2\left\{\frac{1}{6}\Delta^2\psi\right\}\tilde{Z}^2 + O(\mu^3)$$

where the vertical dependence is given by the function

$$\tilde{Z} \equiv \tilde{Z}(z;\eta) = -\frac{1}{2}(\eta - z)^2 + (\eta + 1)(\eta - z).$$

We **choose the ansatz** (in dimensional form) using $Z_0 \equiv 1, \quad Z_n \equiv \tilde{Z}^n \quad (n \geq 1)$

$$\Phi^a = 1\psi + \tilde{Z}\varphi_1 + \tilde{Z}^2\varphi_2 + \cdots + \tilde{Z}^K\varphi_K,$$

and **make** LUKE'S **functional stationary** for functions of this form.

LANNES & BONNETON 2009 Derivation of asymptotic two-dimensional time-dependent equations for surface water wave propagation, Physics of Fluids

Model equations for $K \geq 1$

A family of model equations (2 evolution PDEs & an elliptic system with K unknowns)

$$\partial_t\eta + \nabla\eta\cdot\nabla\psi + (\nabla\eta)^2 H\varphi_1 + H\varphi_1 + H\Delta\psi + L_{0n}\varphi_n = 0,$$

$$\partial_t\psi + g\eta + \frac{1}{2}(\nabla\psi)^2 - \frac{1}{2}\big(1+(\nabla\eta)^2\big)(\varphi_1 H)^2 - (H\Delta\psi + L_{0n}\varphi_n)\,H\varphi_1 = 0,$$

$$(m = 1,\dots,K) \qquad L_{mn}\varphi_n = \zeta_m H^{2m+1}\Delta\psi$$

where for $m = 0$ and $n = 1,\dots,K$,

$$L_{0n} = 2^n \mathrm{B}(n+1)H^{2n+1}\Delta \;+\; 2^{n+1}\mathrm{B}(2n+1)H^{2n}\nabla\eta\cdot\nabla \\ +\; 2^n(2n+1)\mathrm{B}(n+1)\big(2n(\nabla\eta)^2 + H\Delta\eta\big) - \delta_{1n}nH - nH\delta_{1n}(\nabla\eta)^2,$$

where $\delta_{11} = 1$, $\delta_{1n} = 0$ for $n > 1$, and $\mathrm{B}(k) \equiv \mathrm{B}(k,k) = \frac{(k-1)!(k-1)!}{(2k-1)!}$ is the Beta function for equal positive integers. For $m, n = 1,\dots,K$,

$$L_{mn} = \alpha_{mn}H^{2(m+n)+1}\Delta \;+\; \beta_{mn}H^{2(m+n)}\nabla\eta\cdot\nabla \\ +\; \kappa_{mn}H^{2(m+n)-1}\left((\nabla\eta)^2 + \lambda_{mn}H\Delta\eta + \mu_{mn}\right),$$

where α_{mn}, β_{mn}, γ_{mn} are quotient numbers depending only on m and n.

Model equations for $K \geq 1$ (in uniform water depth)

A family of model equations (2 evolution PDEs & an elliptic system with K unknowns)

$$\partial_t \eta + \nabla\eta \cdot \nabla\psi + (\nabla\eta)^2 H\varphi_1 + H\varphi_1 + H\Delta\psi + L_{0n}\varphi_n = 0,$$

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$$(m = 1, \ldots, K) \qquad L_{mn}\varphi_n = \zeta_m H^{2m+1}\Delta\psi$$

❶ Hamiltonian structure:

$$\begin{aligned} \partial_t \eta &= \partial_\psi \mathcal{H}^\alpha \\ \partial_t \psi &= -\partial_\eta \mathcal{H}^\alpha \end{aligned} \quad \text{with} \quad \begin{aligned} \mathcal{H}^\alpha(\eta,\psi) &= \frac{1}{2}\int \left\{ \int_{-h}^{\eta} (\nabla_{X,z}\Phi^\alpha[\eta]\psi)^2 \,\mathrm{d}z + g\eta^2 \right\} \mathrm{d}X \\ &= \frac{1}{2}\int \big(\psi G^\alpha[\eta]\psi + g\eta^2\big)\,\mathrm{d}X \end{aligned}$$

and $\mathcal{H}^\alpha$ positive definite for any K.

- Conserved quantities: $\mathcal{M}^\alpha(t) = \int \eta \,\mathrm{d}X$, $\mathcal{I}_i^\alpha(t) = \int \eta\, \partial_i \psi \,\mathrm{d}X$ and $\mathcal{H}^\alpha$.

❷ Equivalent to the Isobe-Kakinuma (IK) model obtained by choosing

$$\Phi_*^\alpha(X,z,t) = \sum_{n=0}^{K} \phi_n(X,t) Z_n^*(z;h), \quad \text{with} \quad Z_n^*(z;h) = (z+h)^{2n}$$

The $(K=1)$-model (in uniform water depth)

For $K=1$, we have 2 evolution equations & one elliptic PDE

$$\partial_t\eta + \nabla\cdot\left[H\nabla\psi + \frac{1}{3}\nabla(H^3\varphi_1)\right] = 0,$$

$$\partial_t\psi + g\eta + \frac{1}{2}(\nabla\psi)^2 + \frac{1}{2}\left(1+(\nabla\eta)^2\right)(H\varphi_1)^2 - \left[H\Delta\psi + \frac{1}{3}\Delta(H^3\varphi_1)\right]H\varphi_1 = 0,$$

$$\frac{2}{5}H^2\Delta\varphi_1 + 2H\nabla\eta\cdot\nabla\varphi_1 + \left(H\Delta H + (\nabla\eta)^2 - 1\right)\varphi_1 = -\Delta\psi$$

❶ Hamiltonian:

$$\mathcal{H}^\alpha(\eta,\psi) = \frac{1}{2}\int\left\{\left[H\nabla\psi + \frac{1}{3}\nabla(H^3\ell_1[\eta]\psi)\right]\cdot\nabla\psi + g\eta^2\right\}\mathrm{d}X\,.$$

❷ High-order approximation of the water wave equations: $|\mathcal{H}-\mathcal{H}^\alpha| = O(\mu^3)$

Comparison with asymptotic models

★ $(K = 1)$-model $O(\mu^2)$

$$\partial_t \eta + \nabla \cdot \left[H \nabla \psi + \mu \frac{1}{3} \nabla (H^3 \varphi_1) \right] = 0$$

$$\partial_t \psi + \eta + \frac{(\nabla \psi)^2}{2} + \frac{\mu}{2} \left(\mu \, (\nabla \eta)^2 + 1 \right) (H \varphi_1)^2 - \mu \left[H \Delta \psi + \frac{1}{3} \mu \, \Delta (H^3 \varphi_1) \right] H \varphi_1 = 0$$

$$\mu \mathscr{L} \varphi_1 - \varphi_1 + \Delta \psi = 0$$

Comparison with asymptotic models

★ $(K=1)$-model $O(\mu^2)$

$$\partial_t\eta + \nabla\cdot\left[H\nabla\psi + \mu\,\frac{1}{3}\nabla(H^3\varphi_1)\right] = 0$$

$$\partial_t\psi + \eta + \frac{(\nabla\psi)^2}{2} + \frac{\mu}{2}\left(\mu\,(\nabla\eta)^2 + 1\right)(H\varphi_1)^2 - \mu\left[H\Delta\psi + \frac{1}{3}\,\mu\,\Delta(H^3\varphi_1)\right]H\varphi_1 = 0$$

$$\mu\mathscr{L}\varphi_1 - \varphi_1 + \Delta\psi = 0$$

☆ Whitham-Miles-Salmon

$$\partial_t\eta + \nabla\cdot\left[H\nabla\psi + \mu\frac{1}{3}\nabla(H^3\Delta\psi) + \mu^2\frac{1}{3}\nabla(H^3\mathscr{L}\Delta\psi)\right] = 0$$

$$\partial_t\psi + \eta + \frac{1}{2}(\nabla\psi)^2 - \mu\frac{1}{2}(H\Delta\psi)^2 + \mu^2\left[\frac{1}{2}(\nabla H)^2H^2\Delta\psi - \frac{1}{3}\Delta(H^3\Delta\psi)H\Delta\psi\right] = 0$$

with $\mathscr{L} = \mathscr{L}(\eta) = \frac{2}{5}H^2\Delta + 2H\nabla\eta\cdot\nabla + \frac{1}{2}\Delta(H^2)$.

The eqs. accurate to $O(\mu)$ are linearly ill-posed.

Whitham 1967 Variational methods and applications to water waves. Proc. R. Soc. A

Miles & Salmon 1985 Weakly dispersive nonlinear gravity waves. J. Fluid Mech.

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$$\mu\mathscr{L}\varphi_1 - \varphi_1 + \Delta\psi = 0$$

- Serre-Green-Naghdi (SGN) $O(\mu)$

$$\partial_t\eta + \nabla\cdot\left(H\overline{V}\right) = 0$$

$$\partial_t\overline{V}^\star + \nabla\left[\eta + \frac{1}{2}\overline{V}^2 - \mu\left(\frac{1}{3H}\overline{V}\cdot\nabla(H^3\nabla\cdot\overline{V}) + \frac{1}{2}H^2(\nabla\cdot\overline{V})^2\right)\right] = 0$$

$$\overline{V}^\star = \overline{V} - \mu\frac{1}{3H}\nabla(H^3\nabla\cdot\overline{V})$$

Matsuno 2015 Hamiltonian formulation of the extended Green–Naghdi equations. Phys. D: Nonlinear Phenom.

Comparison with asymptotic models

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$$\overline{V}^\star = \overline{V} - \mu \frac{1}{3H} \nabla (H^3 \nabla \cdot \overline{V})$$

The SGN eqs. accurate to $O(\mu^p)$ with p even are linearly ill-posed. At $O(\mu^3)$ they involve 7^{th}-order spatial derivatives.

Matsuno 2015 Hamiltonian formulation of the extended Green–Naghdi equations. Phys. D: Nonlinear Phenom.

Comparison with asymptotic models

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$$\mu \mathscr{L} \varphi_1 - \varphi_1 + \Delta \psi = 0$$

❊ Choi 2022 $O(\mu^2)$

$$\partial_t \eta + \nabla \cdot \left[H \nabla f - \mu \frac{1}{6} H^3 \nabla \Delta f + \mu^2 \frac{1}{120} H^5 \nabla \Delta^2 f \right] = 0$$

$$\partial_t f^\star + \eta + \frac{1}{2} (\nabla f)^2 - \mu \nabla \cdot \left(\frac{H^2}{2} \Delta f \nabla f \right) + \mu^2 \nabla \cdot \left[\frac{H^4}{24} \Delta^2 f \nabla f + \frac{H^4}{16} \nabla (\Delta f)^2 \right] = 0$$

$$f^\star = f - \mu \frac{H^2}{2} \Delta f + \mu^2 \frac{H^4}{24} \Delta^2 f$$

Choi 2022 High-order strongly nonlinear long wave approximation and solitary wave solution. J. Fluid Mech.

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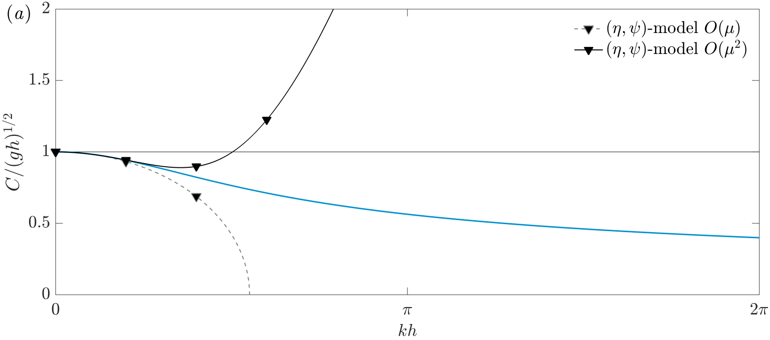
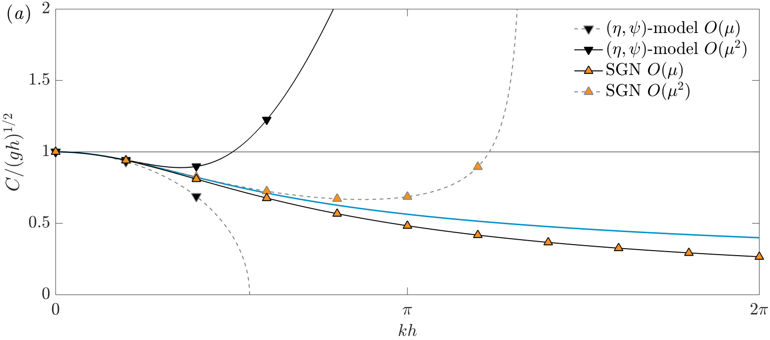
$$f^\star = f - \mu\frac{H^2}{2}\Delta f + \mu^2\frac{H^4}{24}\Delta^2 f$$

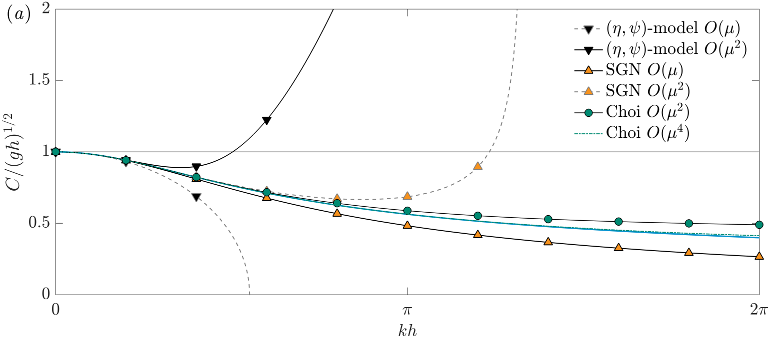
6^{th}-order derivatives. Not exactly Hamiltonian. Unstable when finite-differences are used but stable with a Fourier pseudo-spectral method.

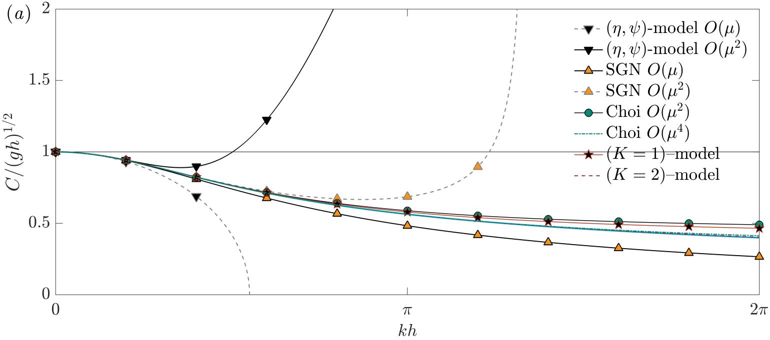
Choi 2022 High-order strongly nonlinear long wave approximation and solitary wave solution. J. Fluid Mech.

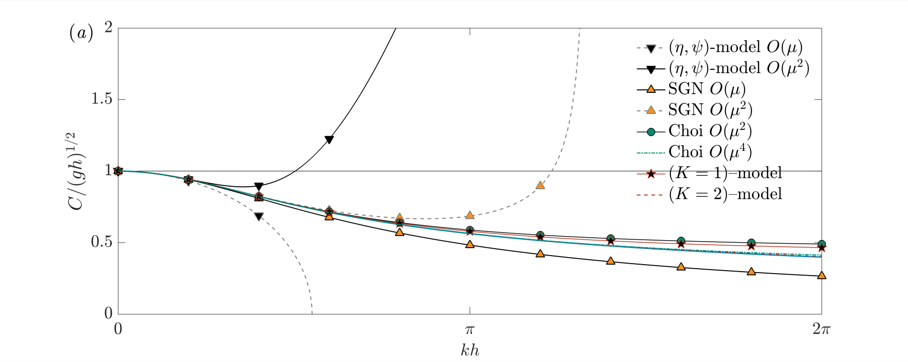

(a)
2
1.5
1
0.5
0
$C/(gh)^{1/2}$
0
π
2π
kh
(η, ψ)-model O(μ)
(η, ψ)-model O(μ²)

Linear Dispersion



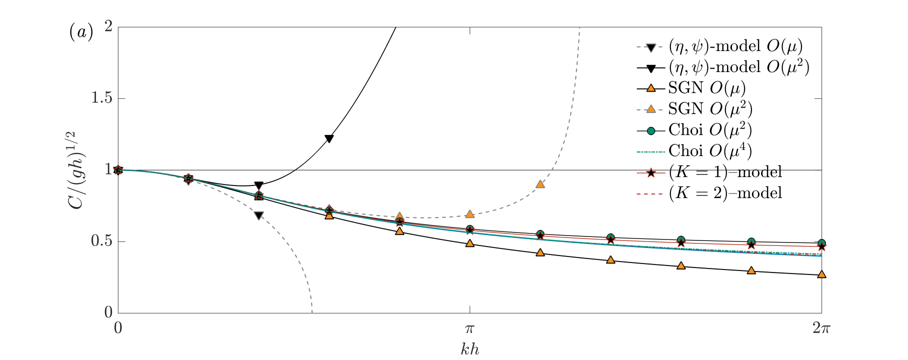

(a)
2
1.5
1
0.5
0
$C/(gh)^{1/2}$
0
π
2π
kh
(η, ψ)-model O(μ)
(η, ψ)-model O(μ²)
SGN O(μ)
SGN O(μ²)
Choi O(μ²)
Choi O(μ⁴)


(a)
2
1.5
1
0.5
0
$C/(gh)^{1/2}$
0
π
2π
kh
(η, ψ)-model $O(\mu)$
(η, ψ)-model $O(\mu^2)$
SGN $O(\mu)$
SGN $O(\mu^2)$
Choi $O(\mu^2)$
Choi $O(\mu^4)$
$(K = 1)$–model
$(K = 2)$–model



❶ $\dfrac{C^2_{(K=1)}}{gh} = \dfrac{1+\frac{1}{15}(kh)^2}{1+\frac{2}{5}(kh)^2}$ is the $[2,2]$–Padé approximant of C^2_{WW}.

❷ For 2% relative error on C_{WW}:
$(K=1)$–model applicable up to $kh \approx 2.84$ (Choi's $O(\mu^2)$ model up to $kh \approx 2.46$).
$(K=2)$–model applicable up to $kh \approx 7.39$ (Choi's $O(\mu^4)$ model up to $kh \approx 5.49$).



The $(K = 1)$-model is free of the trough instability. If $(\eta, \psi) = (\delta h, -\delta h g\, t) + \varepsilon(\tilde{\eta}, \tilde{\psi})\, e^{\mathrm{i}(kx-\omega t)}$, with $\varepsilon \ll 1$ and $\delta \in (-1, 0)$. Then,

$$\frac{C^2}{gh} = \frac{1 + \frac{1}{15}(k(\delta h + h))^2}{1 + \frac{2}{5}(k(\delta h + h))^2}$$

which shows that C^2 is always strictly positive and finite.

Madsen, & Fuhrman 2020 Trough instabilities in Boussinesq formulations for water waves, J. Fluid Mech.

Numerical Methods – (K=1)–model in 1DH dimension

$$\partial_t \eta = -\nabla \cdot \left[H \nabla \psi + \frac{1}{3} \nabla (H^3 \varphi_1) \right] = 0$$

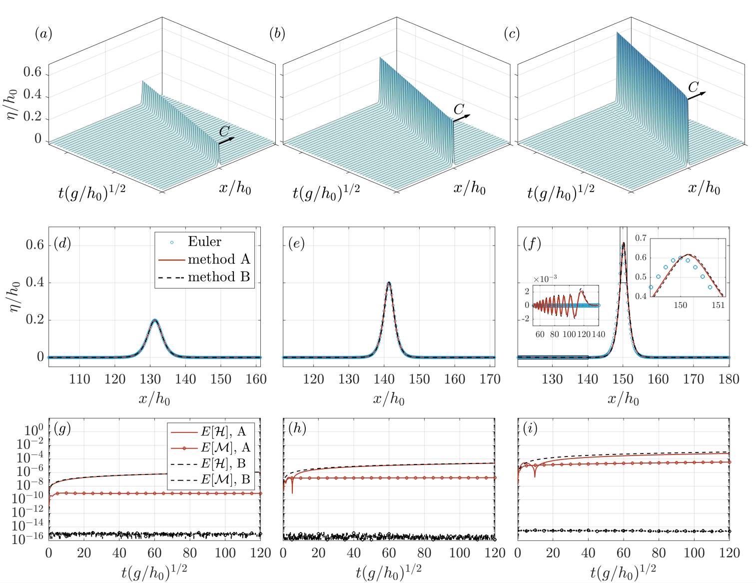
$$\partial_t \psi = -\eta - \frac{(\nabla \psi)^2}{2} - \frac{1}{2} \left(\mu (\nabla \eta)^2 + 1 \right) (H \varphi_1)^2 + \left[H \Delta \psi + \frac{1}{3} \Delta (H^3 \varphi_1) \right] H \varphi_1 = 0$$

$$\mathscr{L} \varphi_1 - \varphi_1 = -\Delta \psi$$

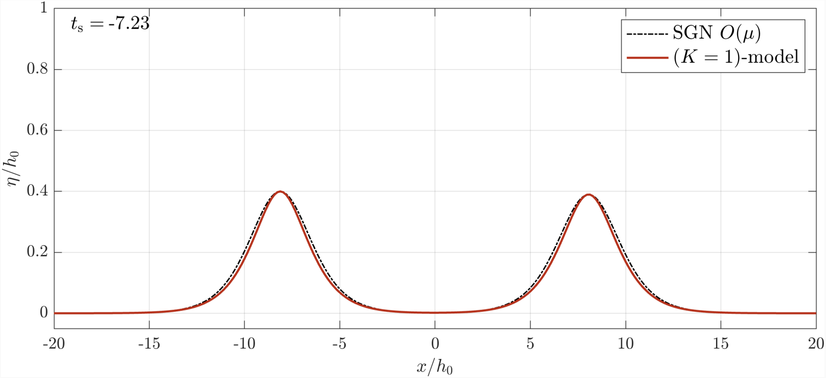
- 1D Spatial discretisation: 4^{th}-order finite differences (method A) or Fourier pseudo-spectral (method B)
- Runge-Kutta 4 in time (with fixed time-step)

Algorithm 1 Time-stepping

```
Initialize (η^0, ψ^0)
for n = 0 to N_t − 1 do
    Compute φ_1 from (η^n, ψ^n)
    Update (η^{n+1}, ψ^{n+1})
end for
```

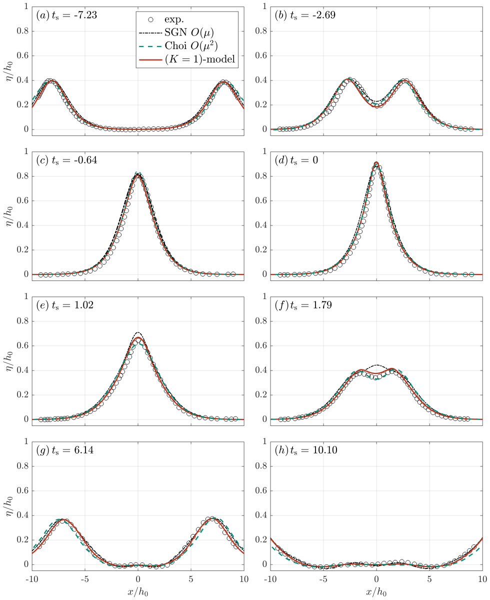

(a)
(b)
(c)
η/h_0
C
$t(g/h_0)^{1/2}$
x/h_0
(d)
(e)
(f)
Euler
method A
method B
x/h_0
(g)
(h)
(i)
$E[\mathcal{H}]$, A
$E[\mathcal{M}]$, A
$E[\mathcal{H}]$, B
$E[\mathcal{M}]$, B
$t(g/h_0)^{1/2}$

Numerical Solutions • Head-on collision of solitary waves

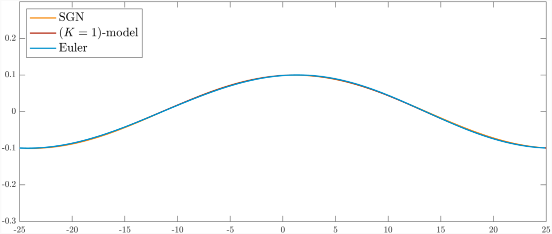


CHEN and YEH 2014, Laboratory experiments on counter-propagating collisions of solitary waves, J. Fluid Mech.

Numerical Solutions • Head-on collision of solitary waves

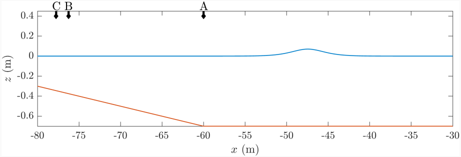


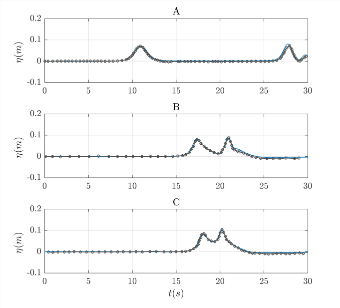
Disintegration of a cos-wave (work in progress)



Zabusky & Kruskal 1965, Interaction of "Solitons" in a Collisionless Plasma and the Recurrence of Initial States, Physical Review Letters

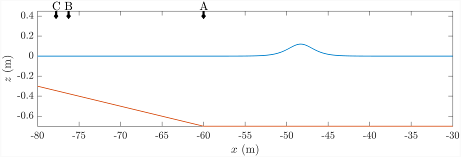
Variable bottom (work in progress)

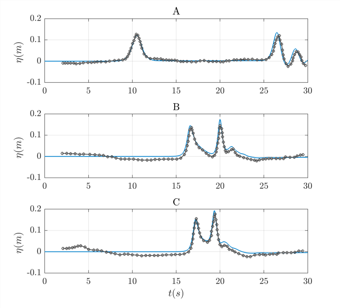




Walkley & Berzins 1999, A finite element method for the one-dimensional extended Boussinesq equations, Int. J. Numer. Meth. Fluids.

Variable bottom (work in progress)





Walkley & Berzins 1999, A finite element method for the one-dimensional extended Boussinesq equations, Int. J. Numer. Meth. Fluids.

Main conclusions

- A new class of model equations for long water waves
- Similar complexity to (high-order) asymptotic models but with 2^{nd} order derivatives
- Exact canonical Hamiltonian structure convenient for numerics
- The solution of one elliptic PDE is needed to advance in time (flat bottom)
- Stable numerical solution with standard methods (Fourier, finite differences)
- Can be extended to higher-order and variable bottom
- No trough instability present in other Boussinesq-type models

Future work

- Waves of permanent form (solitary waves, periodic waves)
- Generating/absorbing boundary conditions, breaking
- Numerical solutions in two horizontal dimensions
- Further experimental validation
- Interfacial waves

Thank you for your attention!