# Wave models under location uncertainty

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with $\overline{u}$ the z-independent horizontal velocity and $h$ the depth of the water column. Here, $\epsilon = A/h$ is the nonlinearity coefficient, and $A$ the typical wave height.

# The location uncertainty formalism (LU)

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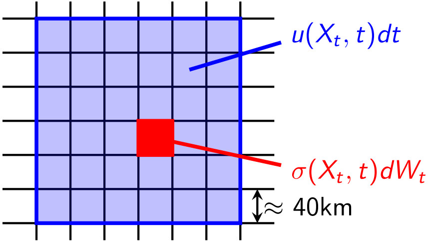

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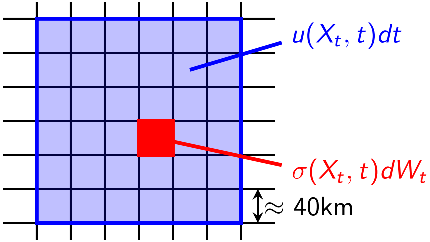


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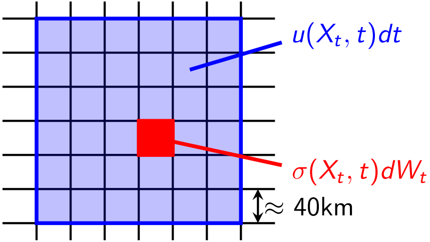


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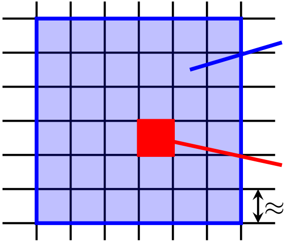


Formally, what is $\sigma dW_t$? $\rightarrow$ **a $Q$-Wiener process**
Let $\hat{\sigma} : [0, T] \rightarrow L^2(\mathcal{S}^2, \mathbb{R}^3)$ a bounded symmetric kernel, then define

$$(\sigma_t f)(x) = \int_{\mathcal{S}} \hat{\sigma}(x, y, t) f(y) dy, \quad \forall f \in L^2(\mathcal{S}, \mathbb{R}^3).$$

Then, define $\sigma_t dW_t = \sum_i \sigma_t e_i \, d\beta^i_t$, and $a(x, t) = \int_{\mathcal{S}} \hat{\sigma}(x, y, t)\hat{\sigma}(y, x, t)dy$, where $(e_i)$ is a basis of $L^2(\mathcal{S}, \mathbb{R}^3)$ and $\beta^i_t$ Brownian motions.

→ New transport operator in the stochastic framework (see [Li21], Chapter 2),

$$\mathbb{D}_t(\cdot) = \partial_t(\cdot) + (u^* \cdot \nabla)(\cdot)dt + \underbrace{(\sigma dW_t \cdot \nabla)(\cdot)}_{\text{transport noise}} - \underbrace{\frac{1}{2}\nabla \cdot (a\nabla(\cdot))}_{\text{stochastic diffusion}}\, dt,$$

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- classical geophysical models [Bau+20; RMC17a; RMC17b; RMC17c]
- stochastic reduced order models [Res+17; Res+21; TCM21]
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- What about waves ???

# Some stochastic wave models

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## LU Serre-Green-Naghdi model

On a spatial domain $\mathcal{S}_H$, with $\mu = h_{ref}^2/L^2$ is the squared aspect ratio,

$$\overline{\mathbb{D}}_t^H h = -\epsilon h \nabla_H \cdot \left(\overline{u}^* dt + \overline{\sigma dW_t}^H\right),$$

$$\overline{\mathbb{D}}_t^H \overline{u} + \nabla_H \eta dt - \frac{1}{h}\epsilon\mu\nabla_H\left(\frac{h^3}{3}(d\overline{G})\right) = \mathcal{O}(\mu^2, \epsilon\mu^2),$$

with $d\overline{G} = \overline{\mathbb{D}}_t^H(\nabla_H \cdot \overline{u}) - \epsilon\left(\nabla_H \cdot \left(\overline{u}^* dt + \overline{\sigma dW_t}^H\right)\right)(\nabla_H \cdot \overline{u}).$

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$$E_{SV} = \frac{\epsilon^2}{2} \int_{\mathcal{S}_H} hu^2 + \frac{1}{2} \int_{\mathcal{S}_H} h^2.$$

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- The deterministic Serre-Green-Naghdi model conserves *another* energy, that is

$$E_{SGN} = \frac{\epsilon^2}{2}\int_{\mathcal{S}_H} hu^2 + \frac{1}{2}\int_{\mathcal{S}_H} h^2 + \frac{\epsilon^3\mu}{6}\int_{\mathcal{S}_H} h^3(\nabla_H\cdot\overline{u})^2.$$

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- The deterministic Boussinesq model does not conserve $E_{SV}$ or $E_{SGN}$, and its LU counterpart does not either under the previous assumption.

# Numerical simulations

# Numerical simulations – Setup

- Periodic boundary conditions on a 2D domain, C-Grid
- Double advection method to simulate the noise terms
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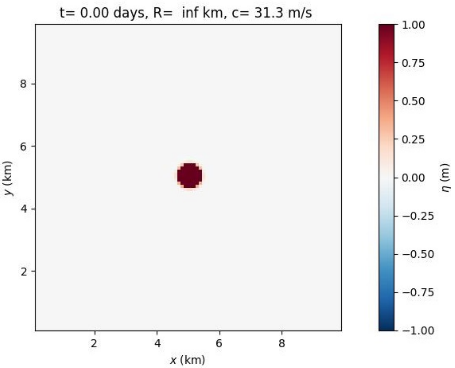


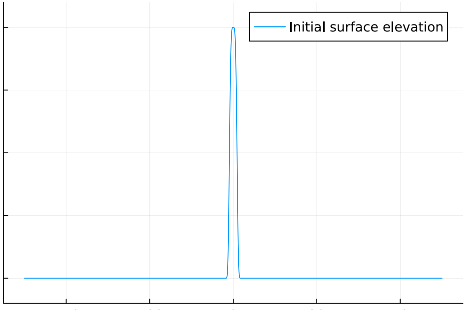

# Numerical simulations – One realisation

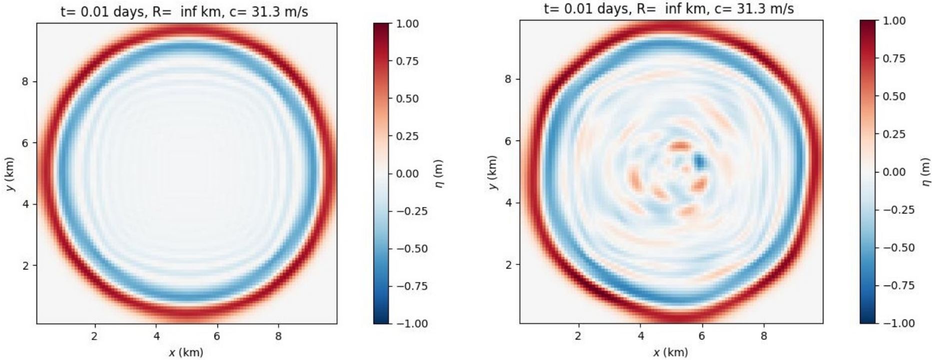


Figure: Deterministic (left) and LU-stochastic (right) Saint-Venant models

Code adapted from the one of Dion Häfner (University of Copenhagen).
NB: Possible in 1D → Vincent Duchêne & Pierre Navarro (Université de Rennes).

# Numerical simulations – Mean and variance (n=100)

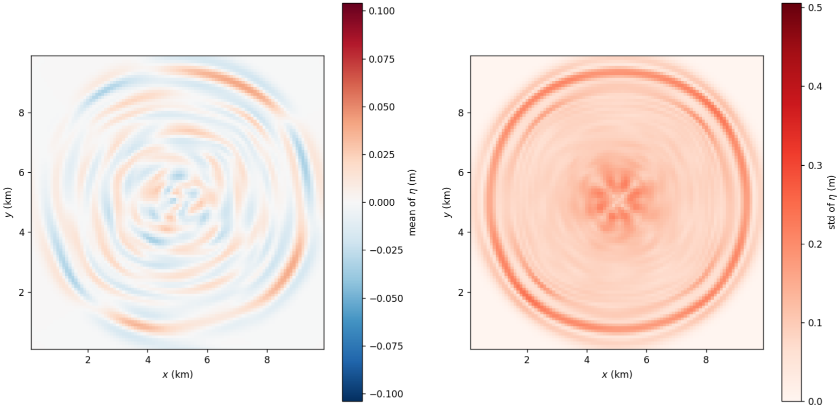


Figure: Difference between the mean of the LU Saint-Venant and the deterministic model (left). Variance of the LU Saint-Venant model (right).

# LU Water Waves model

# LU Water Waves – Derivation I

**Deterministic Water Waves.** Let $\eta$ the surface elevation, $\psi$ the potential at the surface and $G$ the Dirichlet-to-Neumann operator.

$$d_t\eta = G(\eta)\psi,$$
$$d_t\psi + \frac{1}{2}|\nabla\psi|^2 + g\eta - \frac{[G(\eta)\psi + \nabla_H\psi \cdot \nabla_H\eta]^2}{2(1+|\nabla\eta|^2)} = 0.$$

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**LU Water Waves.** Start from the LU Euler's equations with $\nabla \cdot \sigma dW_t = 0$,

$$\mathbb{D}_t u + \frac{1}{\rho_0}\nabla(pdt + dp_t^\sigma) - ge_z dt = 0, \quad \nabla \cdot u = 0.$$

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Moreover, assume $u = (v, w)^{\mathrm{T}} = \nabla\Phi$. The boundary conditions are

- horizontally periodic,
- non penetrating at the (flat) lower boundary (i.e. $w|_{z=-h} = (\partial_z\Phi)|_{z=-h} = 0$),
- free-surface at the upper boundary.

# LU Water Waves – Derivation II

Now, assume $u_s = 0$ for simplicity, so that

$$d_t \nabla \Phi + \left[ \frac{1}{2} \nabla |\nabla \Phi|^2 + \frac{1}{\rho_0} \nabla p - g e_z - u_s \cdot \nabla u - \frac{1}{2} \nabla \cdot (a \nabla u) \right] dt + \sigma dW_t \cdot \nabla u + \frac{1}{\rho_0} \nabla dp_t^{\sigma} = 0.$$

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For small enough noises, stochastic terms are negligible compared to the pressure,

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$\Rightarrow$ Bernoulli equation in the LU stochastic setting

$$d_t \Phi + \frac{1}{2}|\nabla \Phi|^2 dt + g \eta dt = 0.$$

# LU Water Waves – Derivation III

In addition, the kinematic equation at $z = \eta$ reads,

$$d_t\eta + \nabla_H\Phi \cdot \nabla_H\eta dt - \partial_z\Phi dt + \sigma^H dW_t \circ \nabla_H\eta - \sigma^z dW_t = 0,$$

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involving a Stratonovitch transport noise.
Denote by $\psi = \Phi|_{z=\eta}$ the surface potential $\rightarrow$ Dirichlet to Neumann operator $G$

$$G(\eta)\psi = (\partial_z\Phi)|_{z=\eta} - (\nabla_H\Phi)|_{z=\eta} \cdot \nabla_H\eta dt,$$

where $\Phi$ is the solution to

$$\Delta\Phi = 0, \quad \Phi|_{z=\eta} = \psi, \quad (\partial_z\Phi)|_{z=-h} = 0.$$

# LU Water Waves – Derivation III

In addition, the kinematic equation at $z = \eta$ reads,

$$d_t\eta + \nabla_H\Phi \cdot \nabla_H\eta dt - \partial_z\Phi dt + \sigma^H dW_t \circ \nabla_H\eta - \sigma^z dW_t = 0,$$

involving a Stratonovitch transport noise.
Denote by $\psi = \Phi|_{z=\eta}$ the surface potential $\to$ Dirichlet to Neumann operator $G$

$$G(\eta)\psi = (\partial_z\Phi)|_{z=\eta} - (\nabla_H\Phi)|_{z=\eta} \cdot \nabla_H\eta dt,$$

where $\Phi$ is the solution to

$$\Delta\Phi = 0, \quad \Phi|_{z=\eta} = \psi, \quad (\partial_z\Phi)|_{z=-h} = 0.$$

Additionally, denote by

$$F(\eta)\sigma dW_t = \sigma^z dW_t - \sigma^H dW_t \circ \nabla_H\eta,$$

so that

$$d_t\eta = G(\eta)\psi + F(\eta)\sigma dW_t.$$

# LU Water Waves – Comparison to the deterministic system

By inverting a matrix system, we finally have:

**LU Water Waves**

$$d_t\eta = G(\eta)\psi + F(\eta)\sigma dW_t,$$

$$d_t\psi + \frac{1}{2}|\nabla\psi|^2 + g\eta - \frac{[G(\eta)\psi + \nabla_H\psi \cdot \nabla_H\eta]^2}{2(1+|\nabla\eta|^2)} - F(\eta)\sigma dW_t \cdot \frac{[G(\eta)\psi + \nabla_H\eta \cdot \nabla_H\psi]}{1+|\nabla\eta|^2} = 0.$$

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**Deterministic Water Waves**

$$d_t\eta = G(\eta)\psi,$$

$$d_t\psi + \frac{1}{2}|\nabla\psi|^2 + g\eta - \frac{[G(\eta)\psi + \nabla_H\psi\cdot\nabla_H\eta]^2}{2(1+|\nabla\eta|^2)} = 0.$$

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Further work (in progress):

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- Connect the three first LU models to the LU Water Waves one, e.g. as in [Lan13]
- Influence of the (non-flat) bottom? Influence of the Itō-Stokes drift?
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- Water waves without the small noise condition? $\rightarrow$ idea with small space frequency condition

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