Comportement spectral des champs de vagues aléatoires en zone de surf

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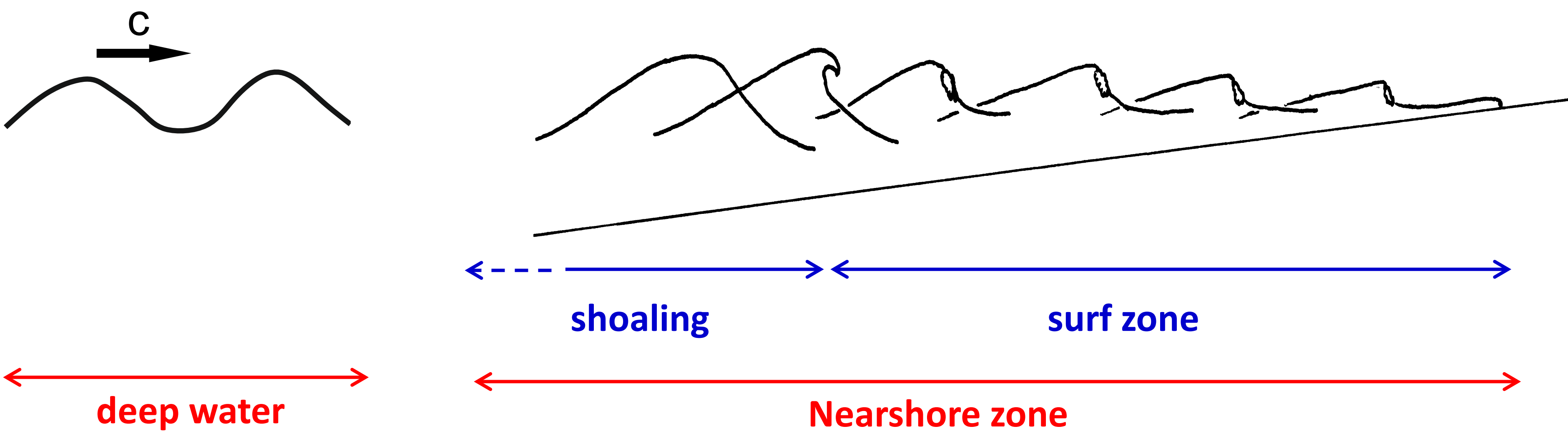
analogy → random broken waves / Burgers turbulence

Wind-generated waves: random state made of a large number of waves which interact nonlinearly $\rightarrow$ **wave turbulence**

- ❑ stochastic description $\rightarrow$ statistical quantities such as energy spectra
- ❑ wave spectrum shape $\rightarrow$ nonlinear and dissipative processes

two regimes of wave turbulence



weakly nonlinear
and strongly dispersive

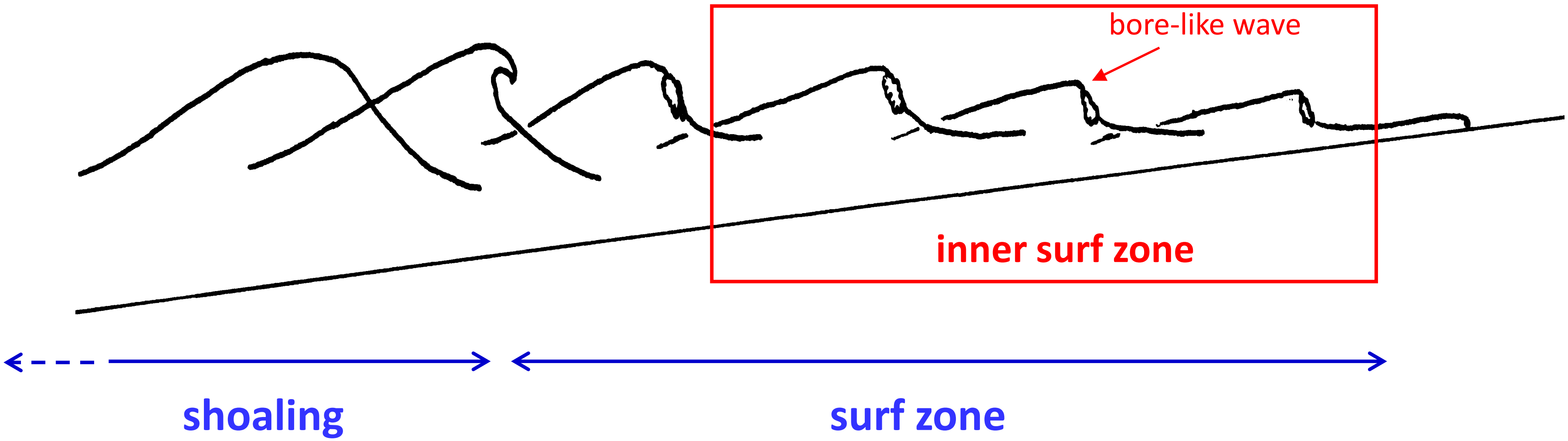
strongly nonlinear
and weakly dispersive

weak wave turbulence

strong wave turbulence

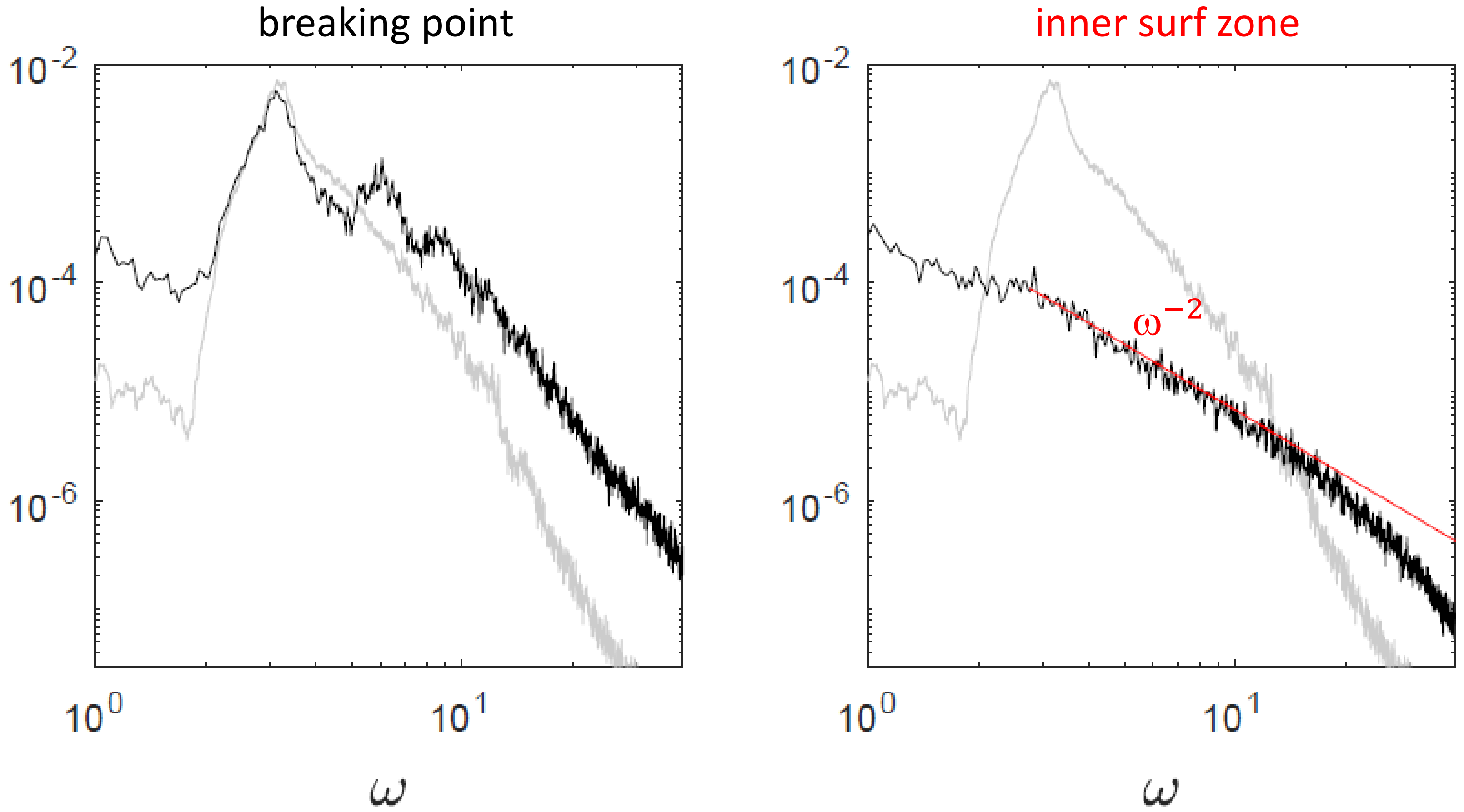
Hasselman, 1962, Zakharov and Filonenko 1966, ….

Nearshore waves



strongly nonlinear and weakly dispersive

Nearshore waves

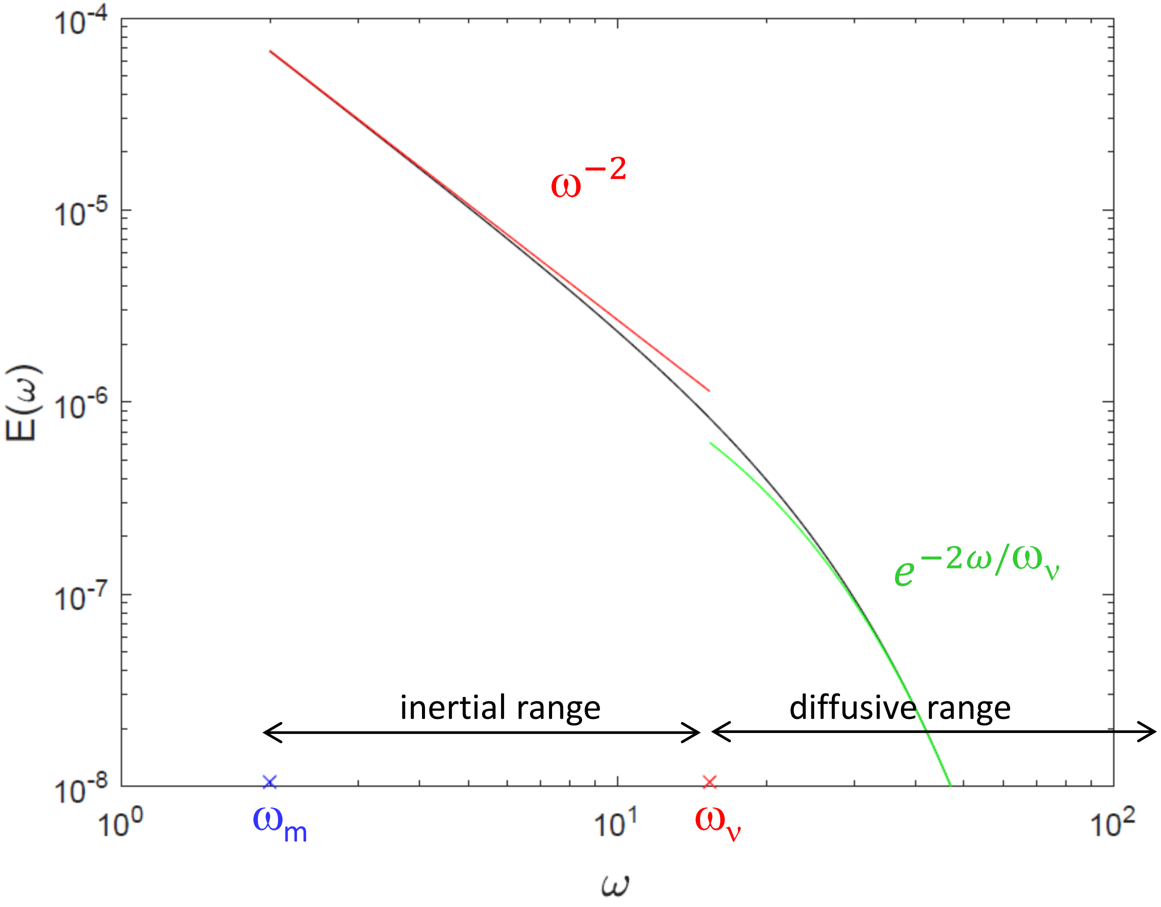


data from van Noorloos, 2003

Inner surf zone spectra follow an universal law

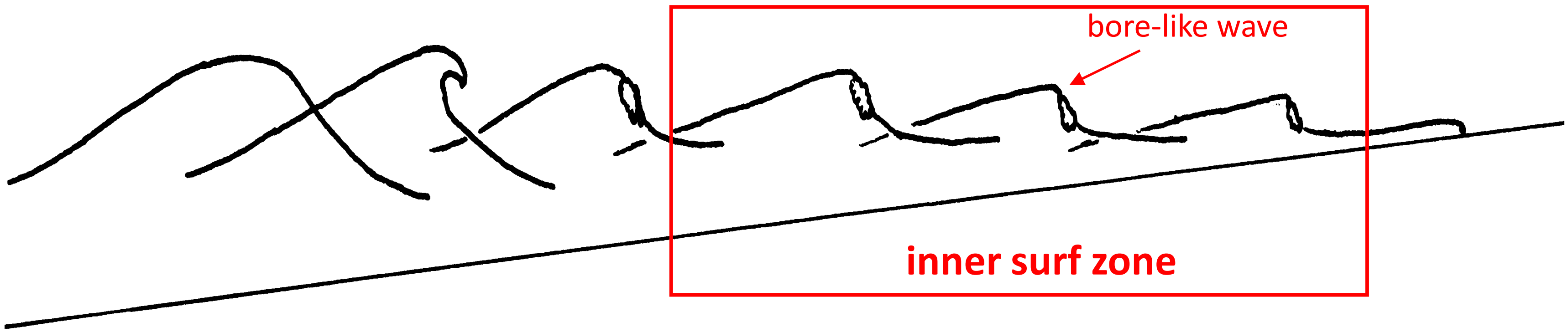
$$E(\omega) = \frac{8}{9}\frac{\nu_c^2}{g}\omega_m \mathrm{csch}^2\left(\frac{\omega}{\omega_\nu}\right)$$

Bonneton, JFM 2023

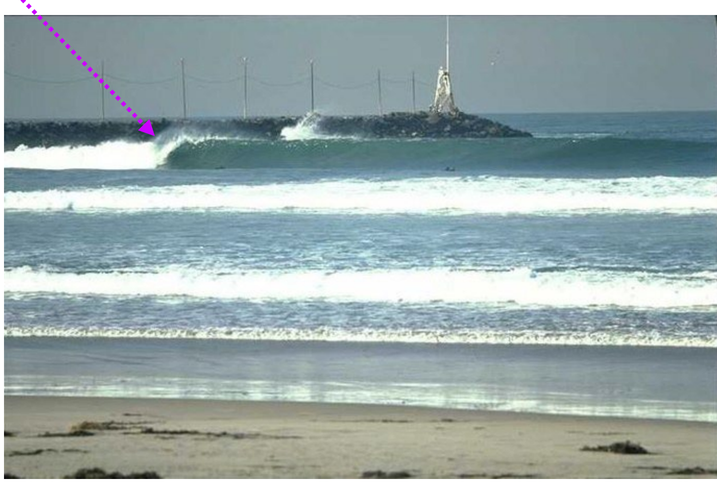


Inner surf zone (ISZ)

Physical background



Breaking onset

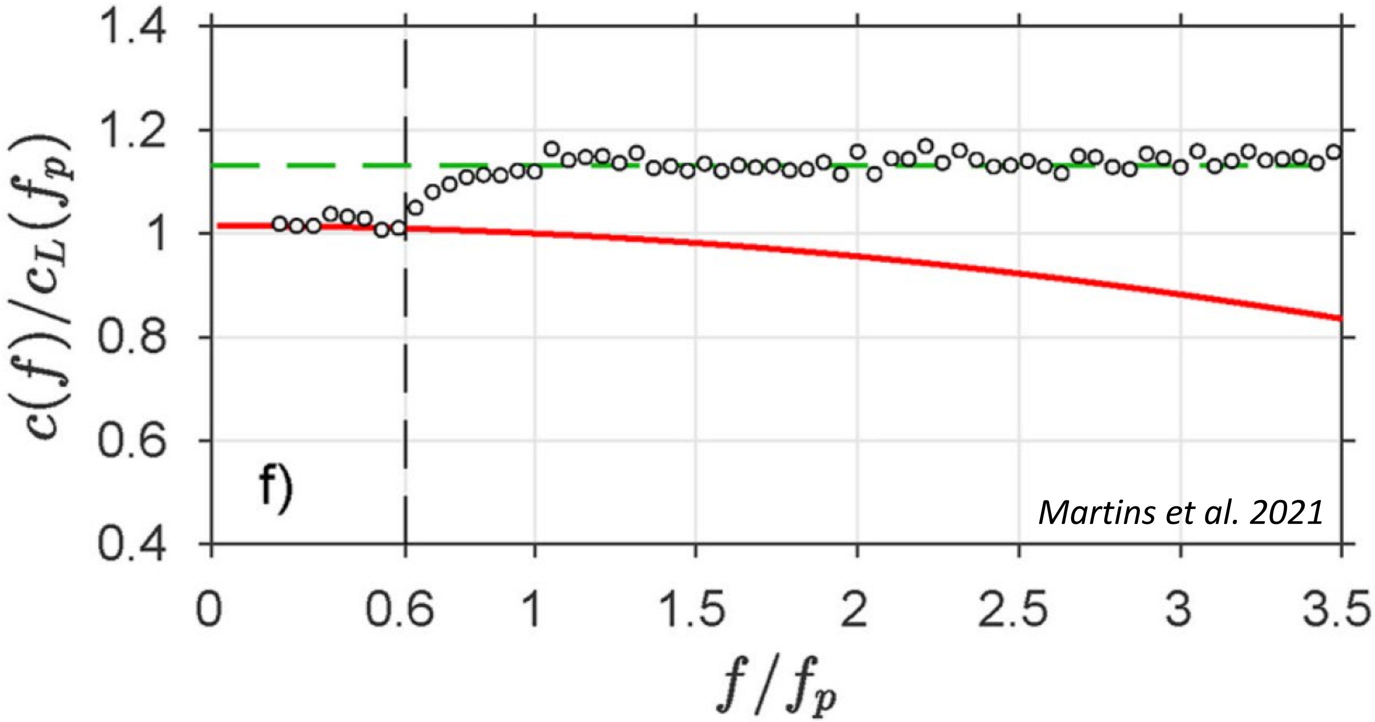


bores

- strongly nonlinear
- dispersive in amplitude

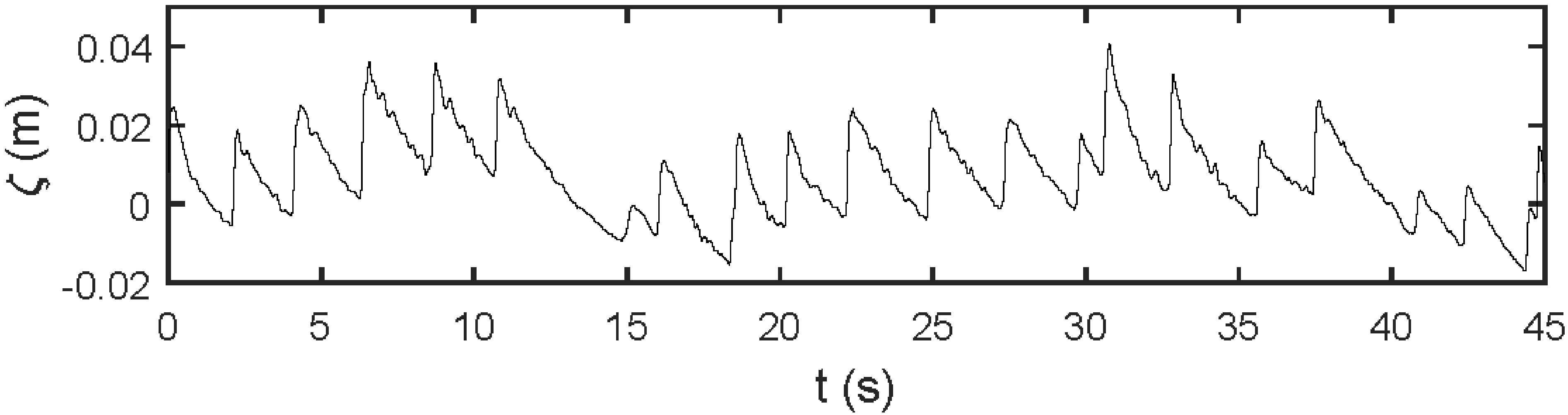
- strongly nonlinear
- dispersive in amplitude
- nearly non-dispersive in frequency

random waves, GLOBEX



$$c_\varphi = \left(\frac{g}{k}\tanh(kh)\right)^{1/2}$$

sawtooth waves (SW)



characteristic feature of nonlinear non-dispersive wave systems

e.g. Burgers' waves or nonlinear acoustic waves

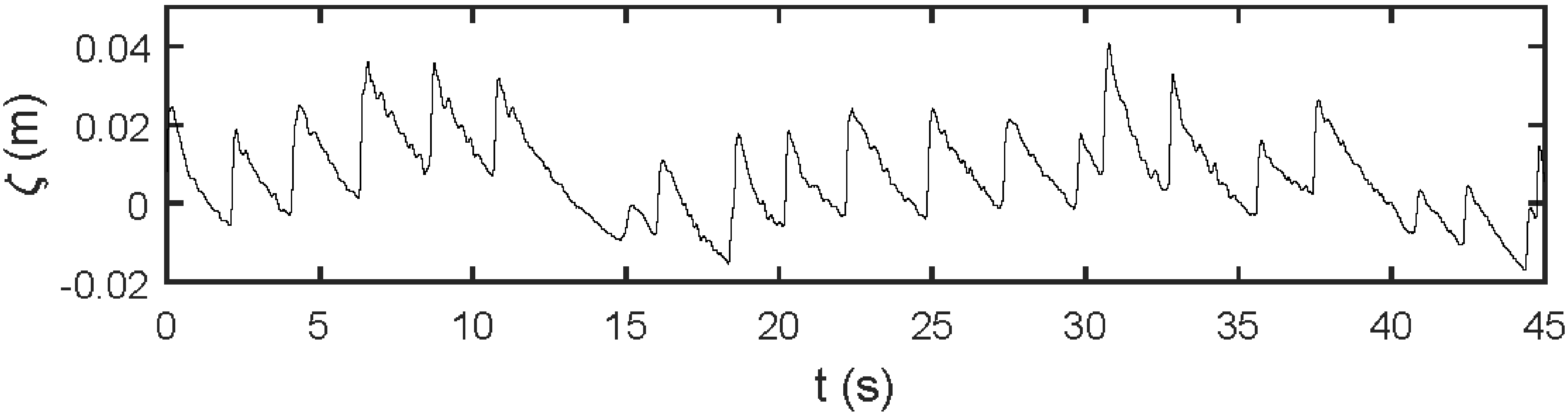
amplitude dispersion

→ **bore merging**

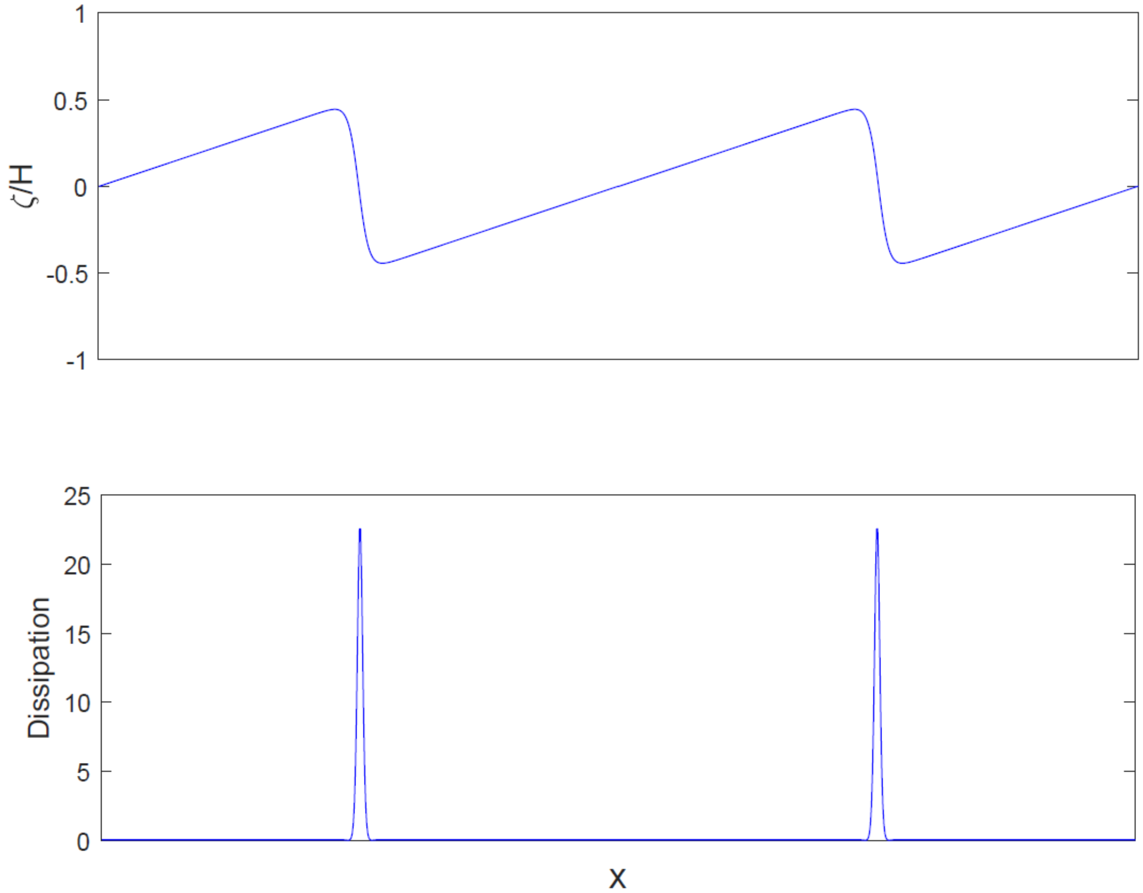
Photos, plage de la Salie, Bonneton 2017

Objective

→ **a better understanding and modelling of the spectral behaviour of random SW in the ISZ**



Physical domain (x,t)



Spectral domain (k, ω)

$E(\omega) \sim \omega^{-2}$

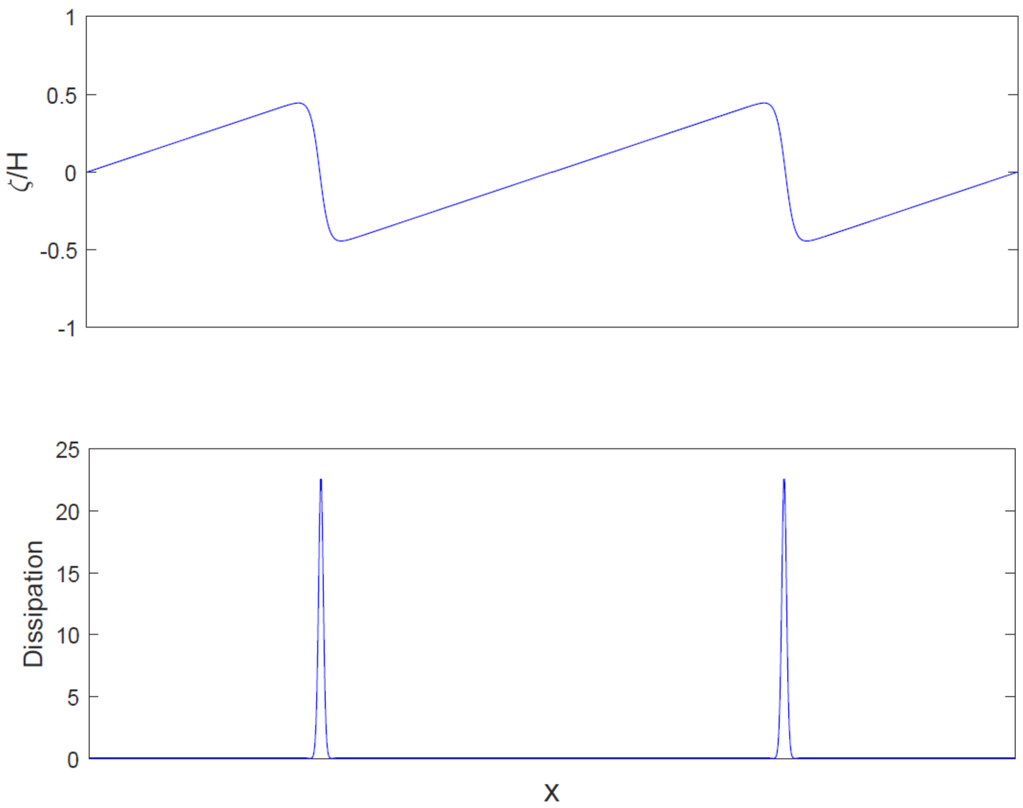
$D(\omega) \sim cst$

"white spectrum"

objective: to go beyond this qualitative description of the spectral behavior

strategy

→ the simplest possible nonlinear shallow water model



mathematical simplicity of the model

→ crucial to derive analytical spectral laws

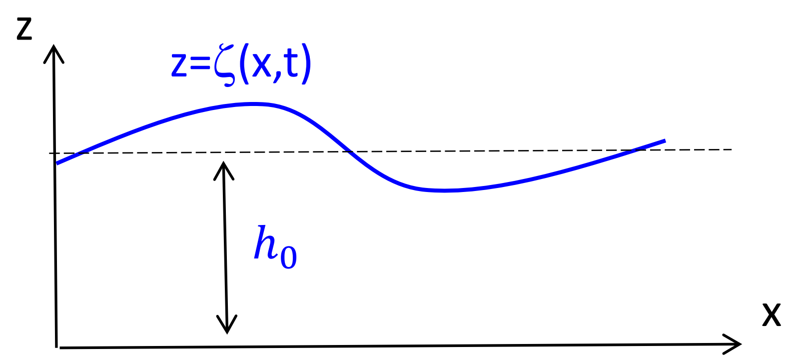
One-way nonlinear non-dispersive shallow water model

$$\frac{\partial \zeta}{\partial t} + c_0 \frac{\partial \zeta}{\partial x} + \frac{3}{2} c_0 \frac{\zeta}{h_0} \frac{\partial \zeta}{\partial x} = -\frac{c_0}{4} \frac{\zeta}{h_0} \frac{dh_0}{\partial x}$$

Bonneton 2007

h_0 : mean water depth

$c_0 = \sqrt{g h_0}$



One-way nonlinear non-dispersive shallow water model

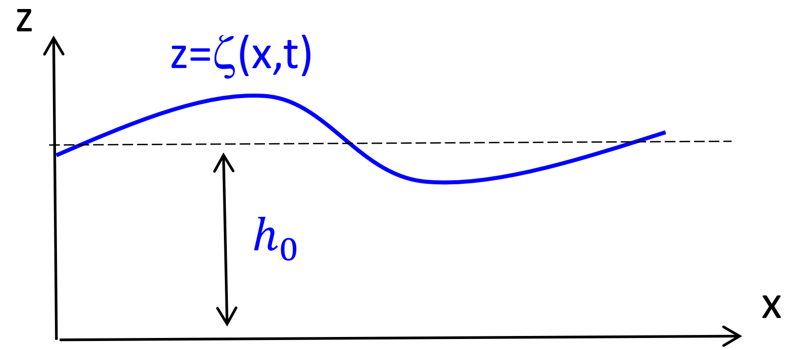
$$\frac{\partial \zeta}{\partial t} + c_0 \frac{\partial \zeta}{\partial x} + \frac{3}{2} c_0 \frac{\zeta}{h_0} \frac{\partial \zeta}{\partial x} = -\frac{c_0}{4} \frac{\zeta}{h_0} \frac{dh_0}{\partial x} + \nu_c \frac{\partial^2 \zeta}{\partial x^2}$$

Bonneton 2007

h_0 : mean water depth

$c_0 = \sqrt{g h_0}$

ν_c : diffusion coefficient



One-way nonlinear non-dispersive shallow water model

$$\frac{\partial \zeta}{\partial t} + c_0 \frac{\partial \zeta}{\partial x} + \frac{3}{2} c_0 \frac{\zeta}{h_0} \frac{\partial \zeta}{\partial x} = \nu_t \frac{\partial^2 \zeta}{\partial x^2}$$

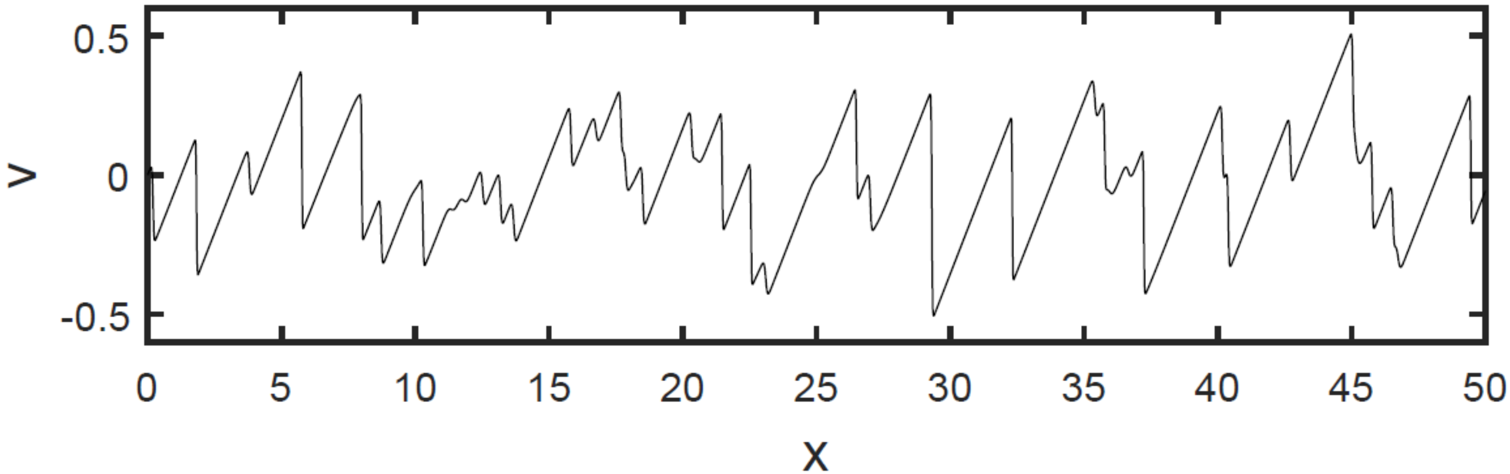
$$v = \frac{3 c_0}{2 h_0} \zeta$$

frame of reference moving at c_0

$$\boxed{\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \nu_t \frac{\partial^2 v}{\partial x^2}}$$

One-way nonlinear non-dispersive shallow water model

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \nu_t \frac{\partial^2 v}{\partial x^2}$$



- nonlinear SW dynamics
- amplitude dispersion
- closely localized dissipation
- bore merging

→ idealized shallow water model to infer the spectral behaviour in the ISZ

→ physical analogy

Spectral behaviour of the Burgers turbulence

Burgers turbulence

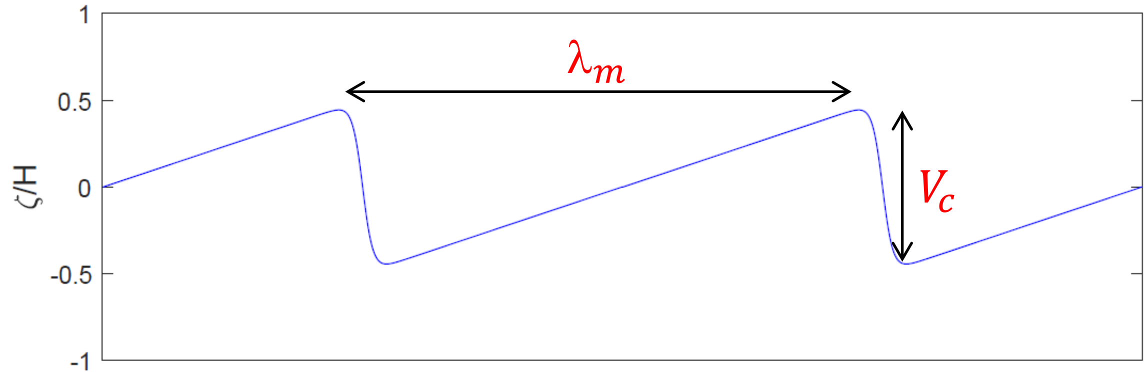
Burgers equation:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \nu\frac{\partial^2 v}{\partial x^2}$$

- Burgers 1948: a simple 1D model to help the understanding of turbulence
- intermittency (shocks) , non-gaussian pdf, energy cascade, … (see Frisch and Bec 2002)
- many physical applications : acoustics, cosmology, plasma physics, vehicle traffic, …
- still an active domain of research

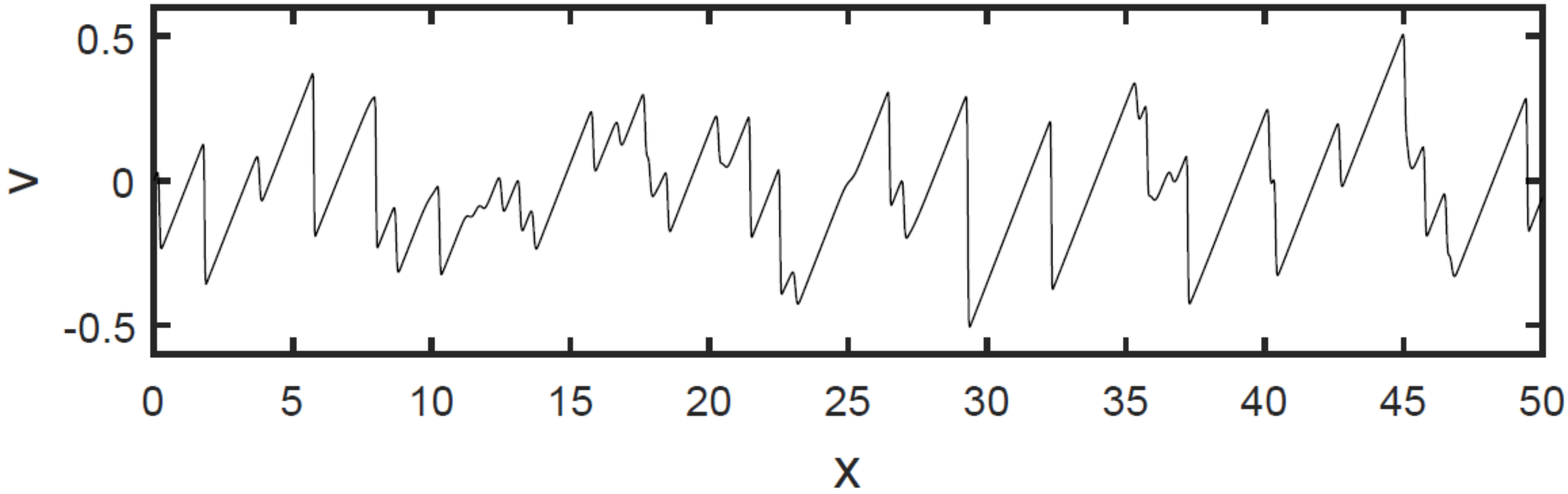
Saffman's work (1968) on the spectral dynamics of random sawtooth waves

Random sawtooth wave field



two length scales: λ_m and $\delta = \nu/V_c$ (average shock thickness)

$$\rightarrow \quad R_B = \frac{V_c \lambda_m}{\nu}$$



freely decaying random solutions, statistically homogeneous in space with <v>=0 power spectral density $E_v(k,t)$

$$E_v(k,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} R(r)\exp(-ikr)dr$$

$$R(r,t) = \langle v(x,t)v(x+r,t)\rangle$$

total energy $\boldsymbol{E}_v = \langle v^2\rangle$

$$\boldsymbol{E}_v = \int_0^{\infty} E_v(k,t)dk$$

$$E_v(k,t) = \frac{2}{\pi k^2}\int_0^{\infty} \frac{d^2Q}{dr^2}\cos(kr)dr$$

where

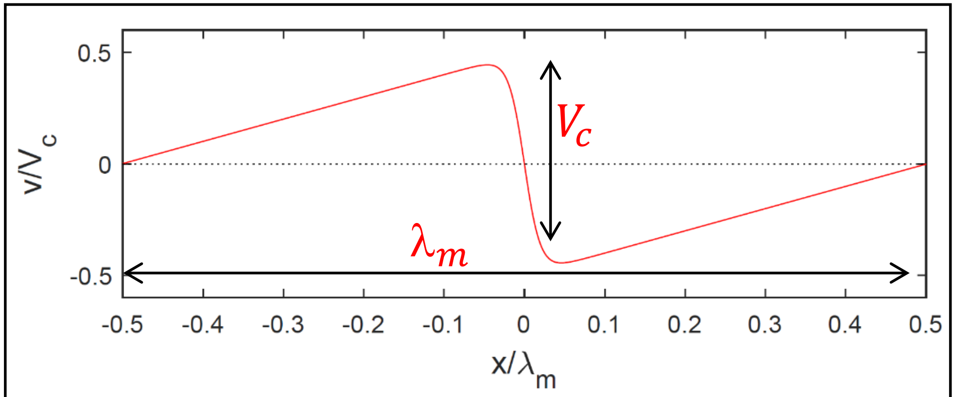
$$Q(r,t) = \frac{1}{2}\langle (v(x+r,t) - v(x,t))^2\rangle$$

second order structure function

Saffman's theory

Assumption:

the periodic SW solution reproduces the qualitative features of the small-scale behavior of random SW



$$v(x,t) = \frac{V_c(t)}{2}\left(\frac{2x}{\lambda_m} - \tanh\frac{V_c(t)x}{4\nu}\right)$$

Khokhlov's solution

For $r \ll \lambda_m$

$$Q(r,t) = \frac{V_c(t)^2}{2\lambda_m(t)}\left(r\coth\left(\frac{V_c(t)}{4\nu}r\right) - \frac{4\nu}{V_c(t)}\right)$$

Saffman's theory

$$Q(r,t) = \frac{V_c(t)^2}{2\lambda_m(t)}\left(r\,\coth\left(\frac{V_c(t)}{4\nu}r\right) - \frac{4\nu}{V_c(t)}\right)$$

$$E_v(k,t) = \frac{2}{\pi k^2}\int_0^\infty \frac{d^2Q}{dr^2}\cos(kr)dr$$

$$\boxed{E_v(k) = \nu^2 k_m \mathrm{csch}^2(\pi\nu k/(2V_c))}$$

$$k_m = \frac{2\pi}{\lambda_m}$$

Saffman's theory

$$Q(r,t) = \frac{V_c(t)^2}{2\lambda_m(t)} \left(r \coth\left(\frac{V_c(t)}{4\nu} r \right) - \frac{4\nu}{V_c(t)} \right)$$

$$E_v(k,t) = \frac{2}{\pi k^2} \int_0^\infty \frac{d^2 Q}{dr^2} \cos(kr) dr$$

$$E_v(k) = \nu^2 k_m \text{csch}^2(\pi \nu k/(2V_c))$$

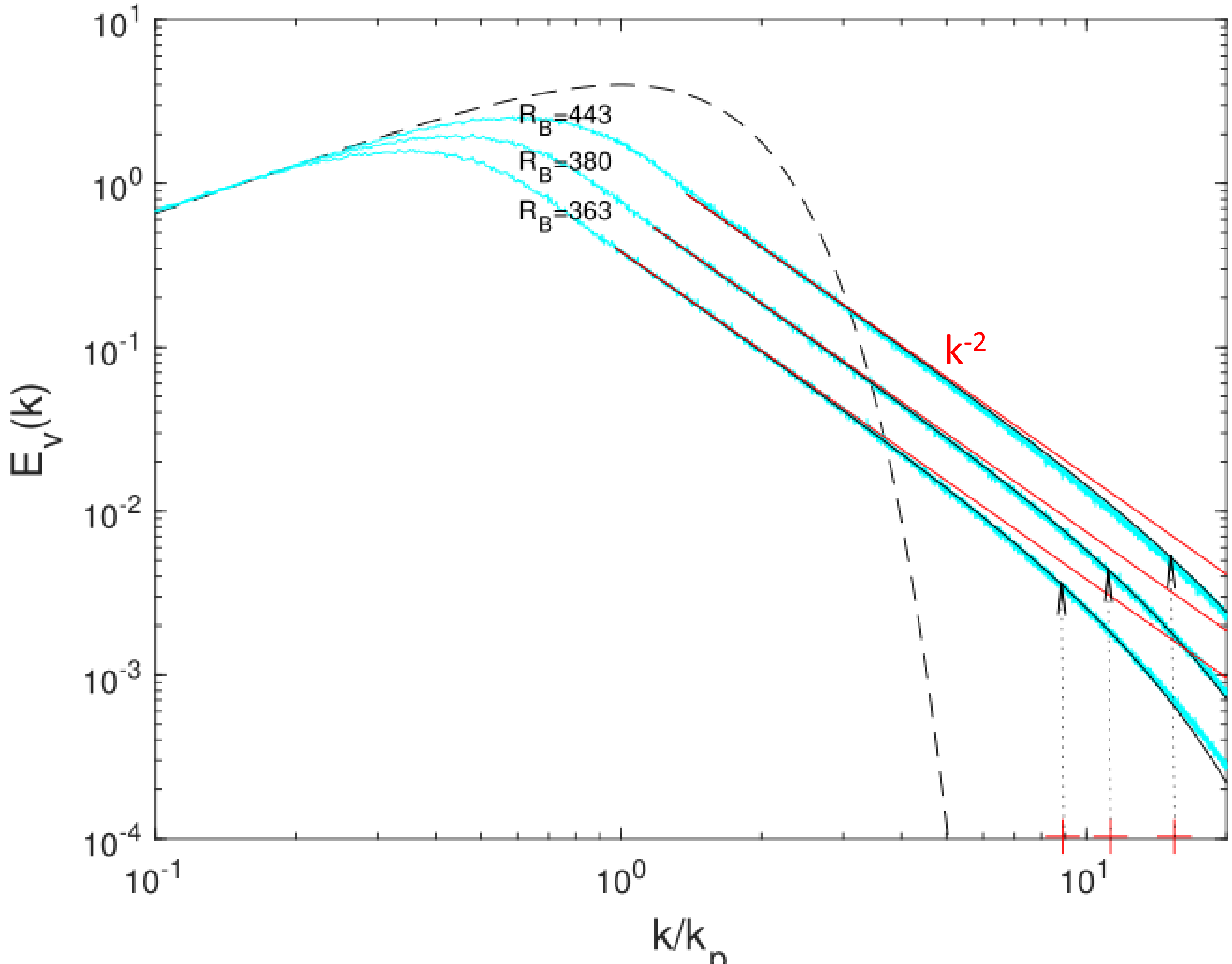
$$\boxed{E_v(k) = 2\nu^2 k_m \text{csch}^2(2\pi \nu k/V_c)} \;*$$

* *Thanks to Jean-François Bony for the help with the integral*

First numerical validation of Saffman's theory

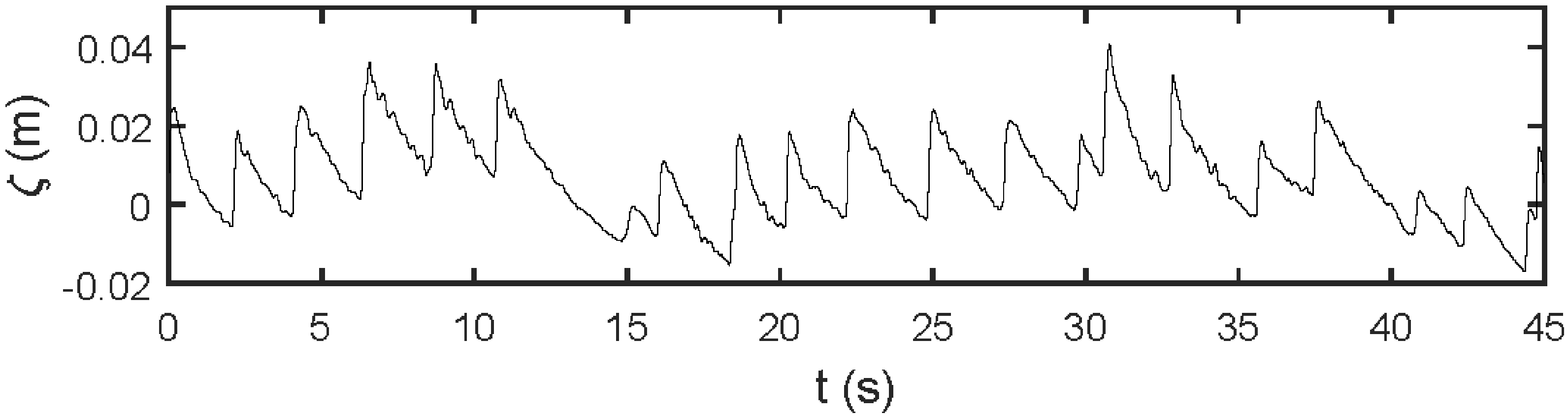
- spectral numerical method
- initial random condition $v_0(k)$ are specified by
  - $$E_0(k) = E_0(k_p)\frac{k}{k_p}\exp\left(-\frac{1}{2}\left((k/k_p)^2 - 1\right)\right)$$
  - a random phase is assigned to each k
- $E_v(k)$ is estimated by an ensemble average over 1000 realizations

$$E_v(k,t) = 2\nu^2 k_m(t)\mathrm{csch}^2\left(\frac{k}{k_\nu(t)}\right)$$



Spectral behaviour of random waves in the ISZ

Analytical energy spectrum for ISZ waves



Hypothesis: $E_v(k)$, in the ISZ, has a shape similar to the one derived from the Burgers-type nonlinear shallow water model

$$\frac{\partial \zeta}{\partial t} + c_0 \frac{\partial \zeta}{\partial x} + \frac{3}{2} c_0 \frac{\zeta}{h_0} \frac{\partial \zeta}{\partial x} = \nu_t \frac{\partial^2 \zeta}{\partial x^2}$$

somewhat exploratory $\rightarrow$ convincing results

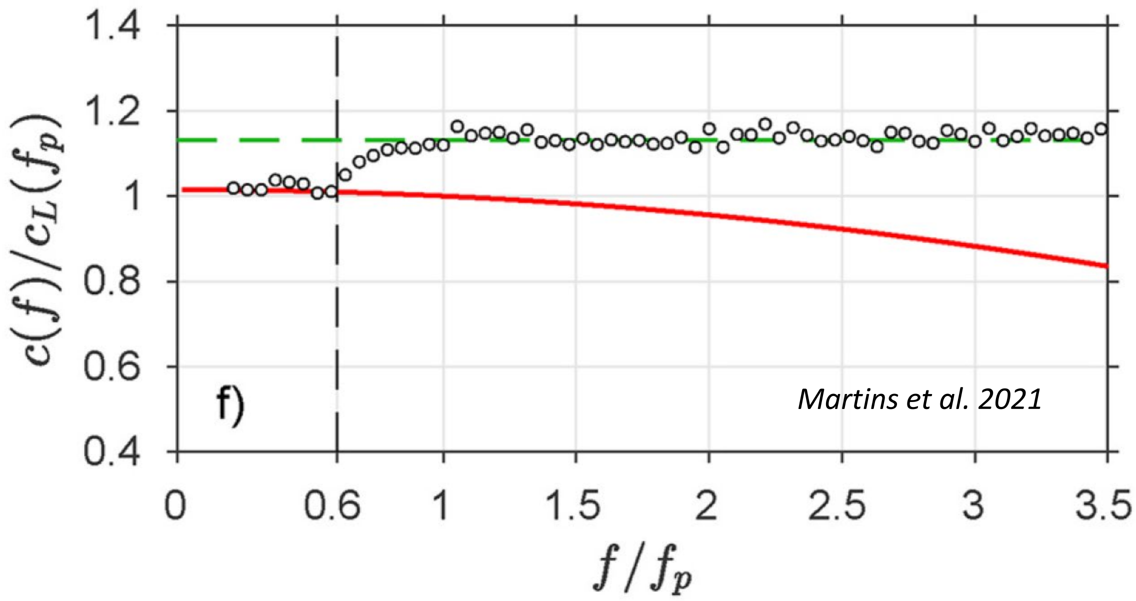
Analytical energy spectrum for ISZ waves

- Spectrum in the frequency domain

$$E_v(k) \longrightarrow E_v(\omega)$$

$$\omega = c_m k \qquad c_m = \alpha_c c_0 \qquad \alpha_c \simeq 1.14$$

$$E_v(\omega) = \frac{2\nu_c^2 \omega_m}{c_0^2} \mathrm{csch}^2\left(\frac{\omega}{\omega_\nu}\right)$$



$$c_0 = \sqrt{g h_0}$$

Analytical energy spectrum for ISZ waves

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$$E_v(\omega) = \frac{2\nu_c^2 \omega_m}{c_0^2} \operatorname{csch}^2\left(\frac{\omega}{\omega_\nu}\right)$$

- Physical variable ζ

$$\zeta = \frac{2h_0}{3c_0} v$$

$$E(\omega) = \frac{8}{9}\frac{\nu_c^2}{g}\omega_m \operatorname{csch}^2\left(\frac{\omega}{\omega_\nu}\right)$$

Analytical energy spectrum for ISZ waves

$$E(\omega) = \frac{8}{9}\frac{\nu_c^2}{g}\omega_m \mathrm{csch}^2\left(\frac{\omega}{\omega_\nu}\right) \qquad \omega \in [\omega_m, \infty]$$

$$\omega_m = 2\pi/T_m \ , \quad \nu_c \ , \quad \omega_\nu = \frac{3}{4\pi}\frac{gH_c}{\nu_c}$$

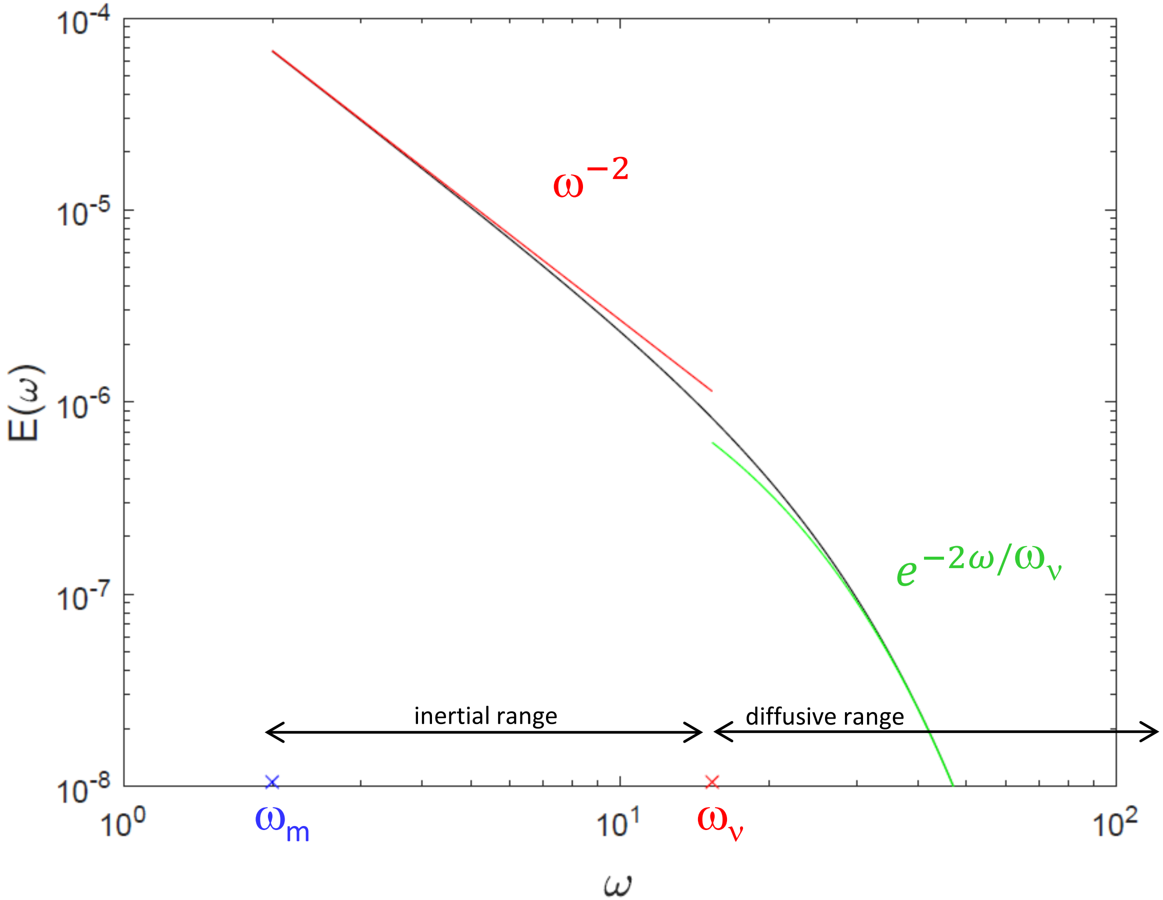
- $\omega \ll \omega_\nu$

$$E(\omega) = \left(\frac{g\omega_m H_c^2}{2\pi^2}\right)\omega^{-2}$$

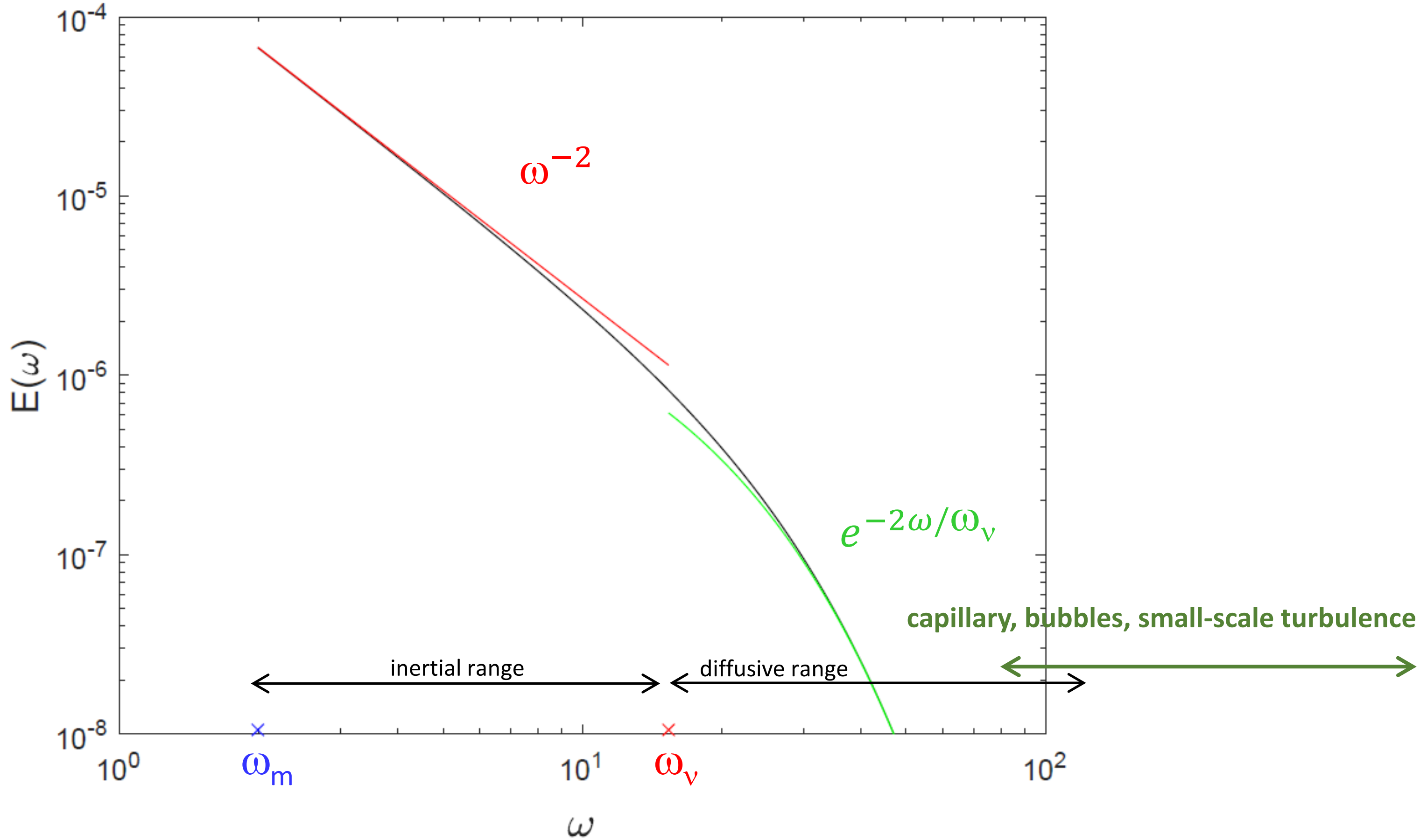
- $\omega \gg \omega_\nu$

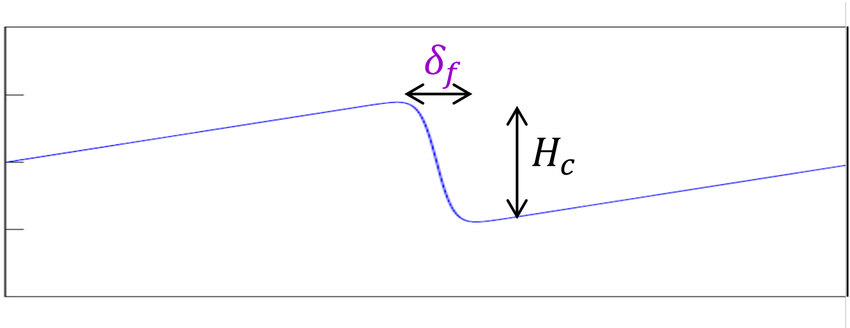
$$E(\omega) = \frac{32}{9}\frac{\nu_c^2}{g}\omega_m \exp\left(-2\frac{\omega}{\omega_\nu}\right)$$

Analytical energy spectrum for ISZ waves



Analytical energy spectrum for ISZ waves





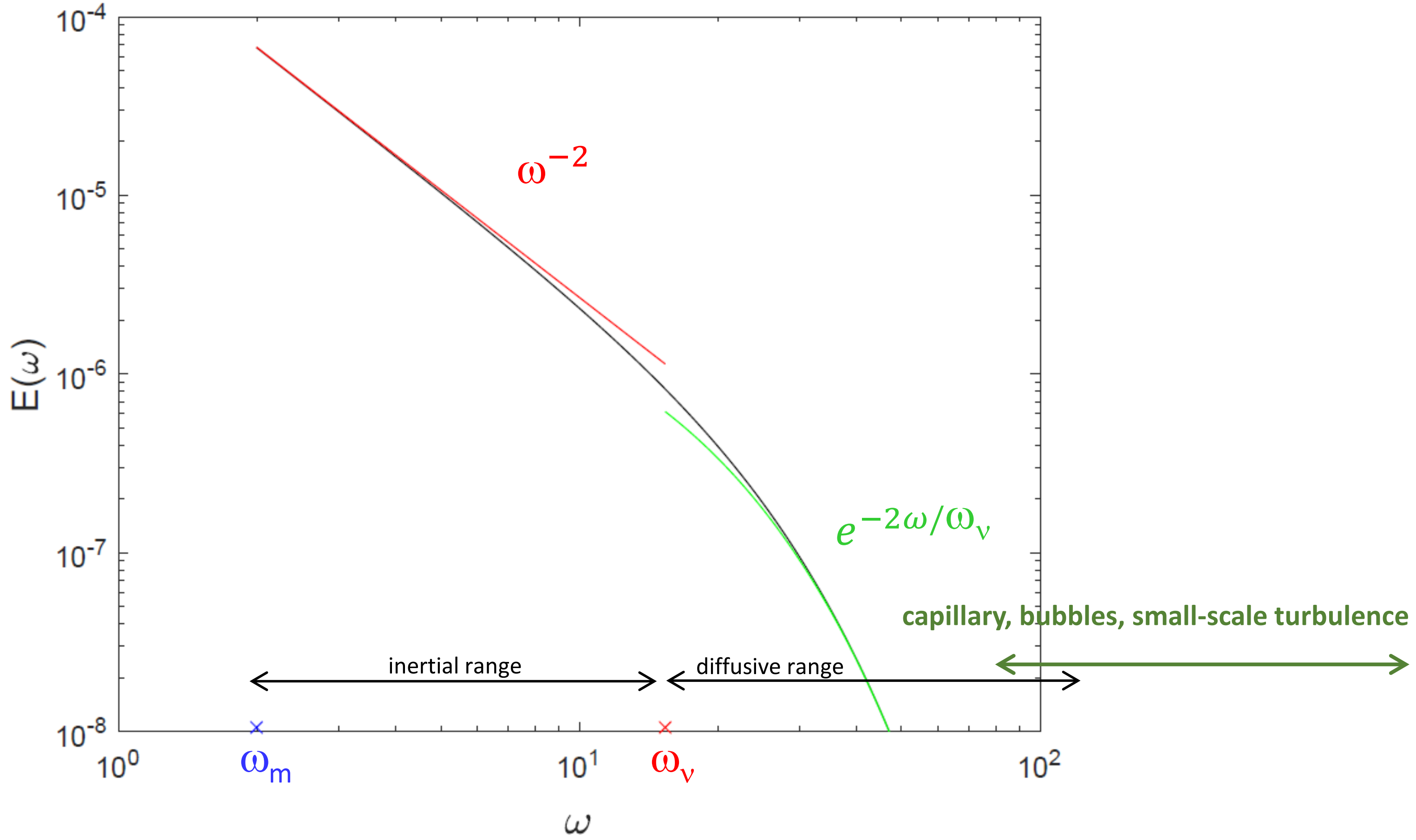
$$l_\nu \approx \pi\, \delta_f$$

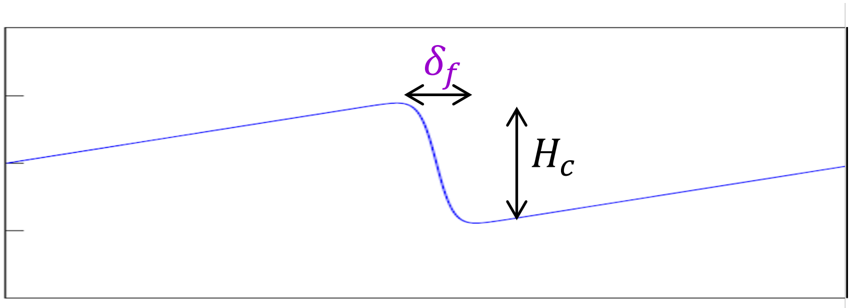
<u>in the field</u>:

$\delta_f > 1\, m$ $\gg$ capillary, bubble and

Kolmogorov scales

Analytical energy spectrum for ISZ waves

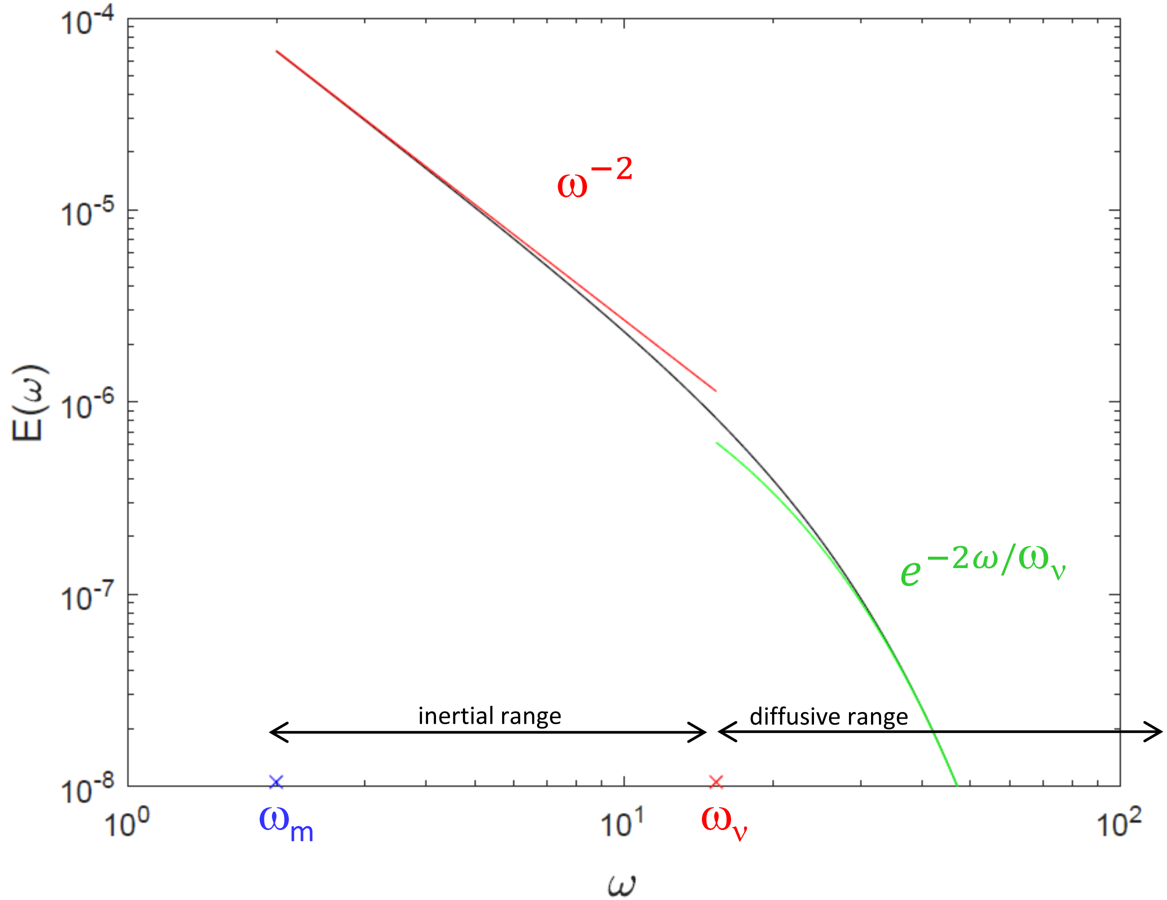




<u>diffusive coefficient</u> $\nu_c(x)$

- not a truly turbulent diffusive coefficient
- incorporates the diffusive effects of various high frequency processes

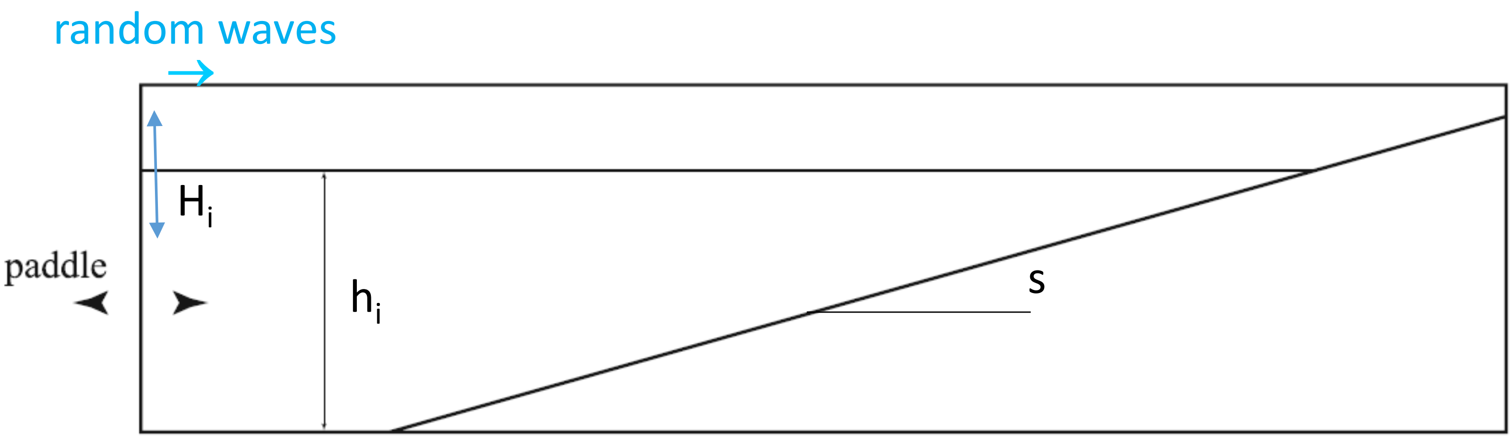
Analytical energy spectrum for ISZ waves



$$\frac{\omega_\nu}{\omega_m} \quad \text{or} \quad R_B = 4\pi^2 \frac{\omega_\nu}{\omega_m} = \frac{3}{2}\frac{gH_c T_m}{\nu_c} \sim \frac{\text{nonlinearity}}{\text{diffusivity}}$$

Application to laboratory experiments

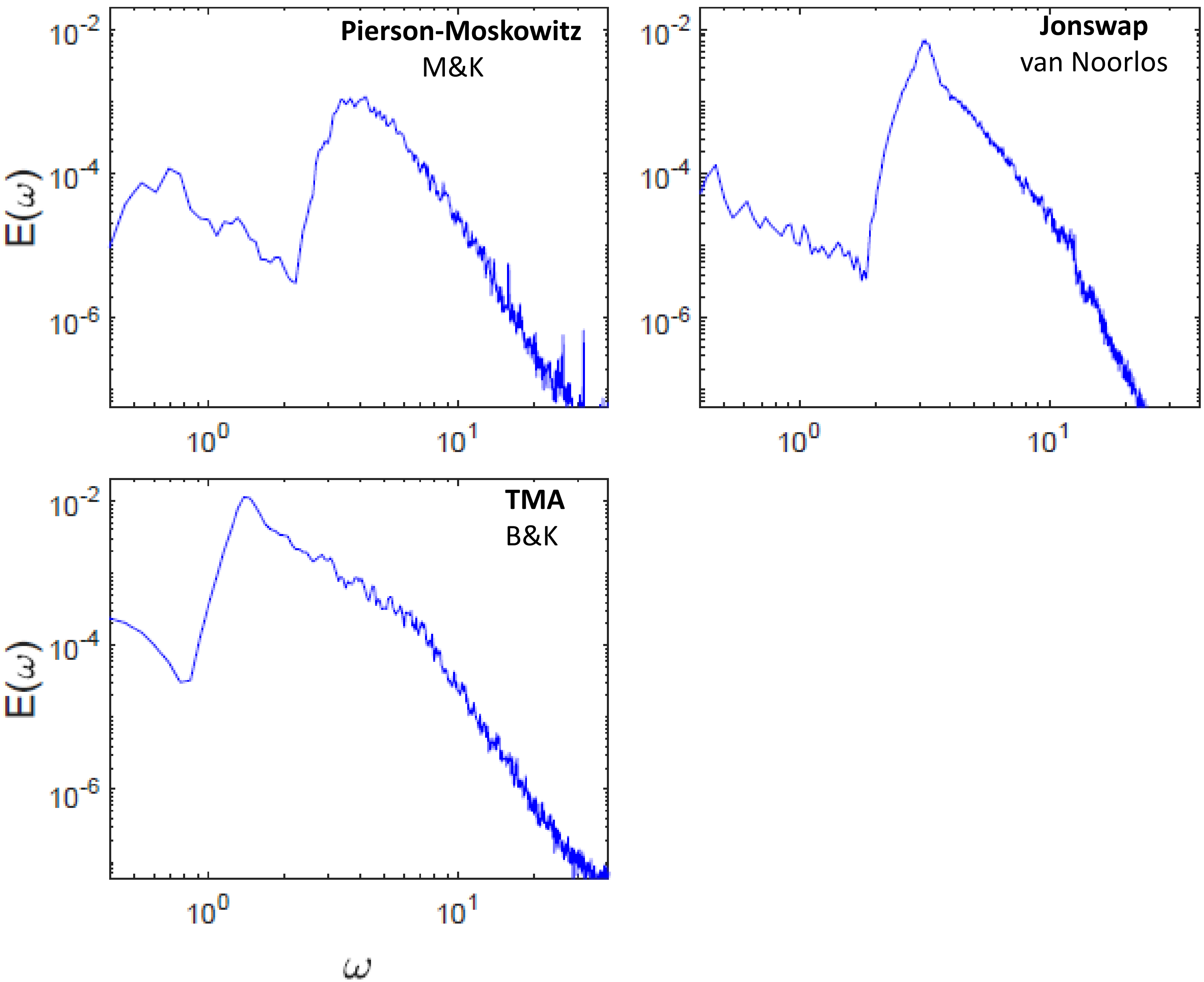
Mase and Kirby 1993, Bowen and Kirby 1994, van Noorloos 2003



	H_i (cm)	h_i (cm)	f_p (Hz)	spectrum	γ	s
vN03-C3	$H_{m0} = 10$	70	0.5	Jonswap	3.3	1/35
vN03-D3	$H_{m0} = 10$	70	0.65	Jonswap	3.3	1/35
MK93	$H_{rms} = 4$	47	0.6	Pierson-Moskowitz	1	1/20
BK94-7	$H_{rms} = 7$	44	0.5	TMA	3.3	1/35
BK94-8	$H_{rms} = 8$	44	0.225	TMA	3.3	1/35
BK94-9	$H_{rms} = 9$	44	0.225	TMA	3.3	1/35

Table 1: Experimental parameters. H_i, incident wave height; h_i, water depth; f_p, peak frequency; γ, peak-enhancement factor; s, bottom slope.

Incident wave spectra



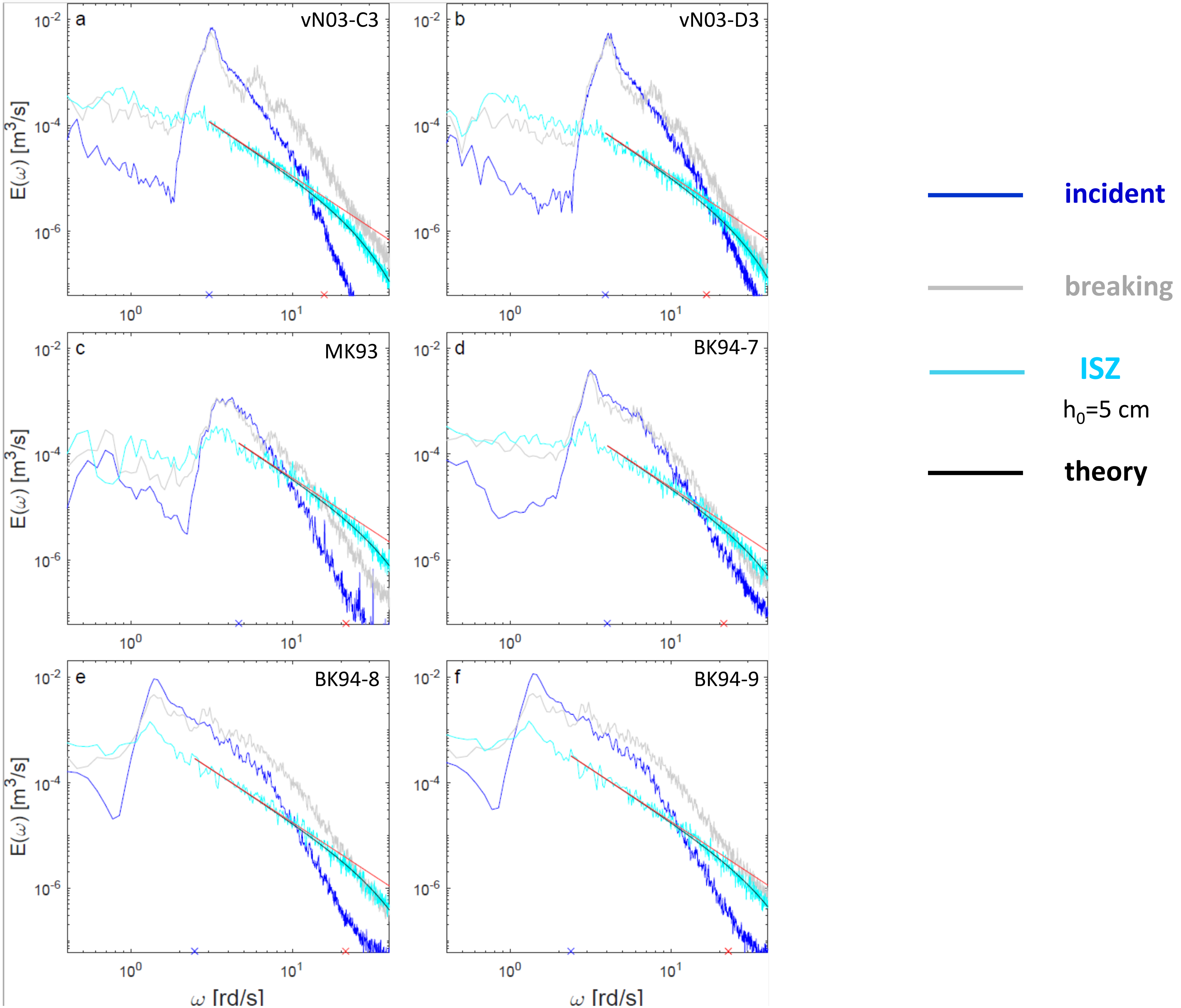
Application to laboratory experiments

$$E(\omega) = \frac{8}{9}\frac{\nu_c^2}{g}\omega_m \mathrm{csch}^2\left(\frac{\omega}{\omega_\nu}\right) \qquad \omega \in [\omega_m, \infty]$$

3 parameters : $(\omega_m, \ \omega_\nu, \ \nu_c)$ or $(\omega_m, \ \omega_\nu, \ \tilde{\mathrm{E}})$

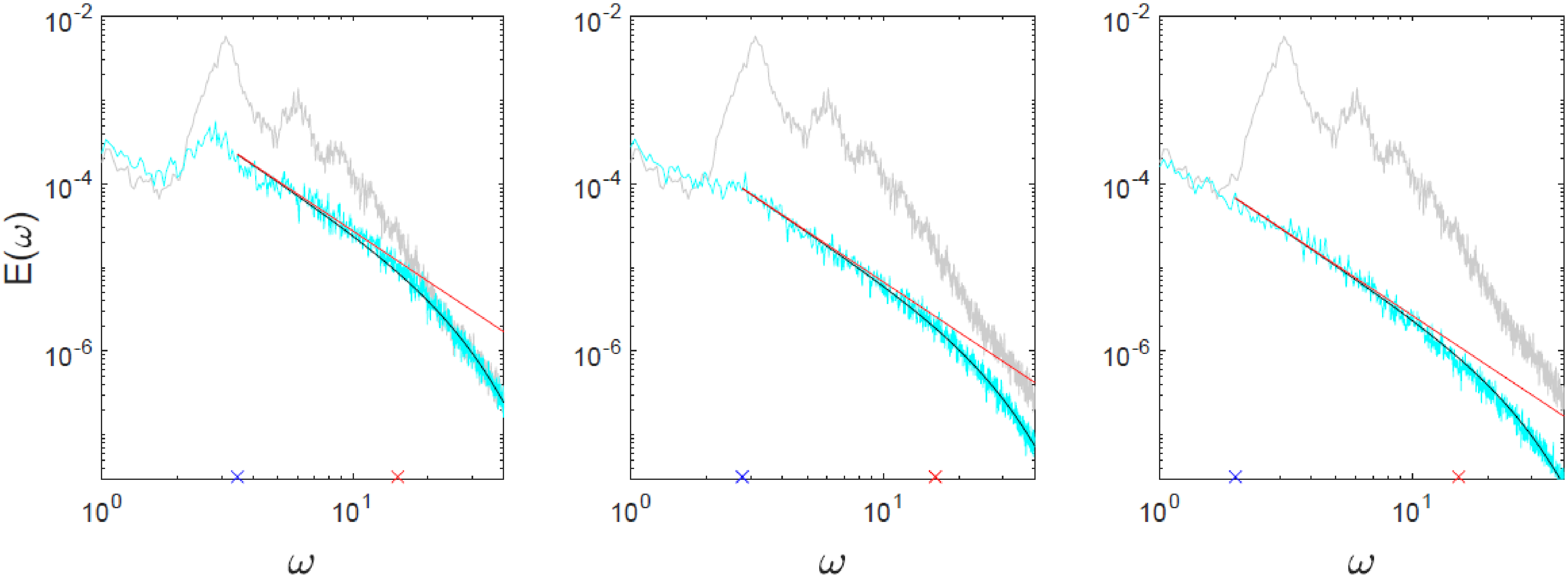
$$\tilde{\mathrm{E}} = \int_{\omega_m}^{\infty} E(\omega)d\omega = \frac{8}{9}\frac{\nu_c^2}{g}\omega_m\omega_\nu\left(\coth(\omega_m/\omega_\nu) - 1\right)$$

1. wave-by-wave analysis $\longrightarrow \omega_m = 2\pi/T_m$
2. calculate $\tilde{\mathrm{E}}$
3. ω_ν unknown of the problem $\longrightarrow$ non-linear least squares method



a
vN03-C3
b
vN03-D3
c
MK93
d
BK94-7
e
BK94-8
f
BK94-9
E(ω) [m³/s]
ω [rd/s]
incident
breaking
ISZ
h0=5 cm
theory

Measurements at different positions in the ISZ

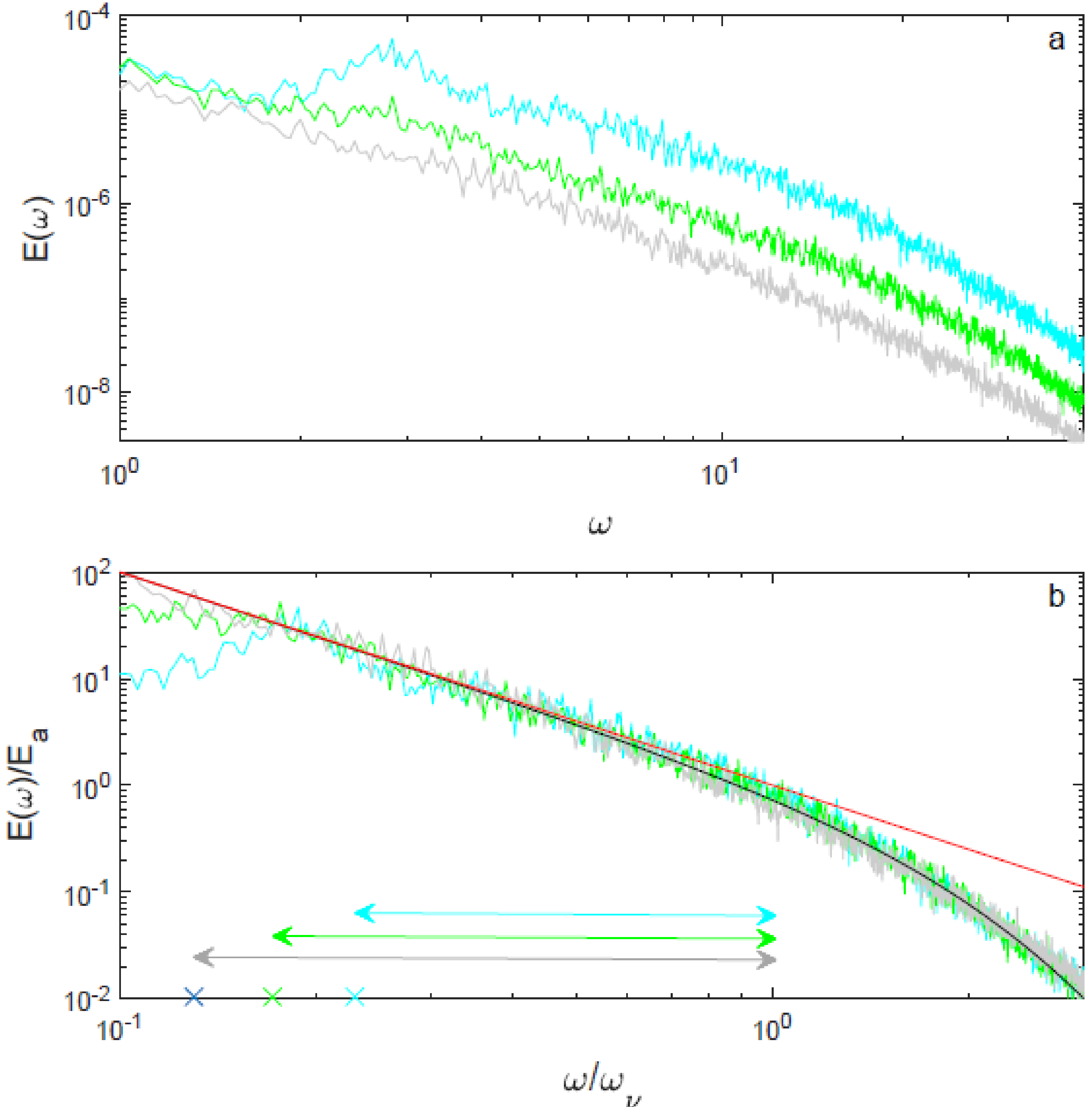


$$R_B = 4\pi^2 \frac{\omega_\nu}{\omega_m}$$

data: van Noorloos 2003

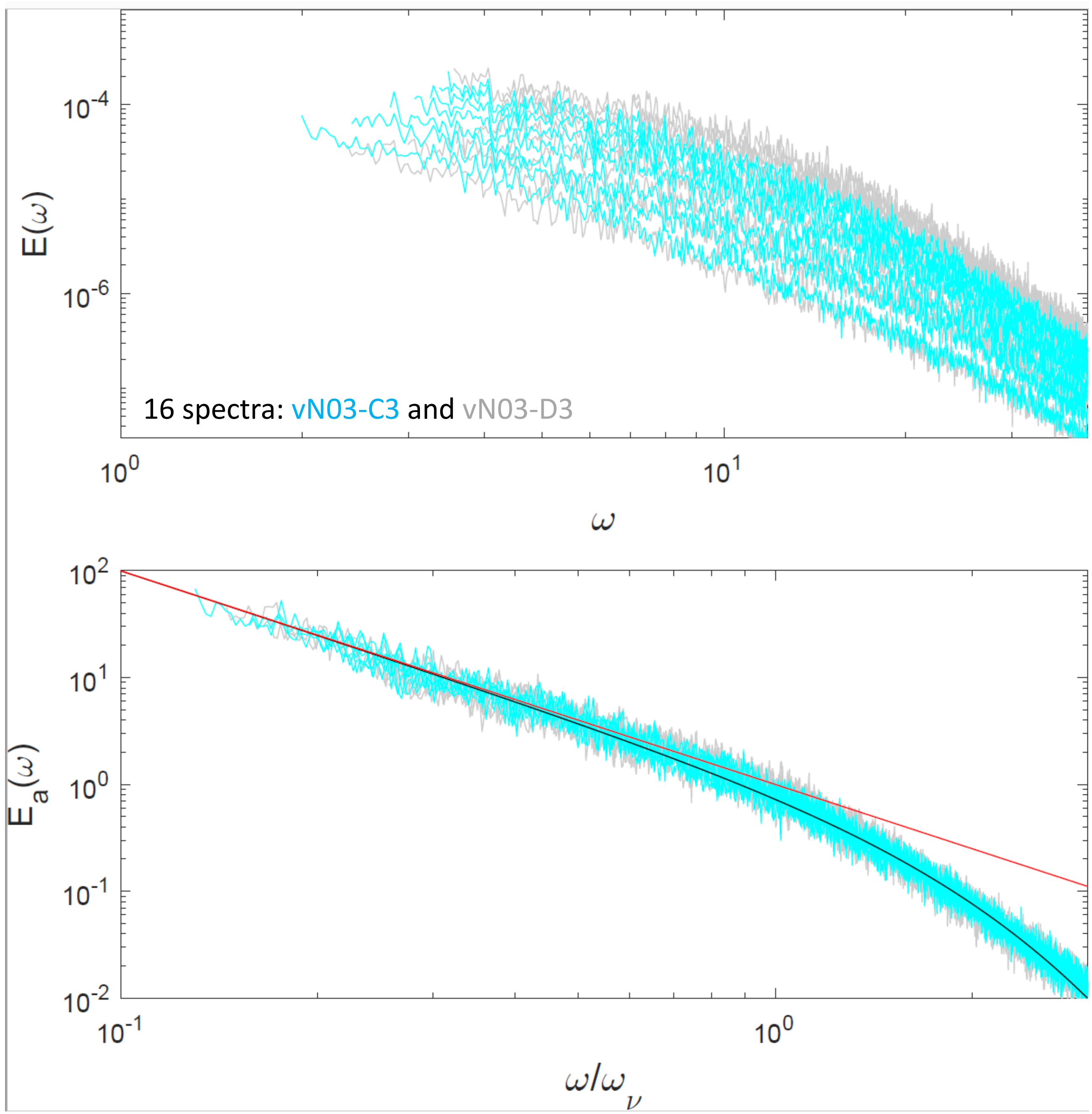
$$\frac{E}{E_a} = \operatorname{csch}^2\left(\frac{\omega}{\omega_\nu}\right)$$

$$E_a = \frac{8\nu_c^2\omega_m}{9g}$$



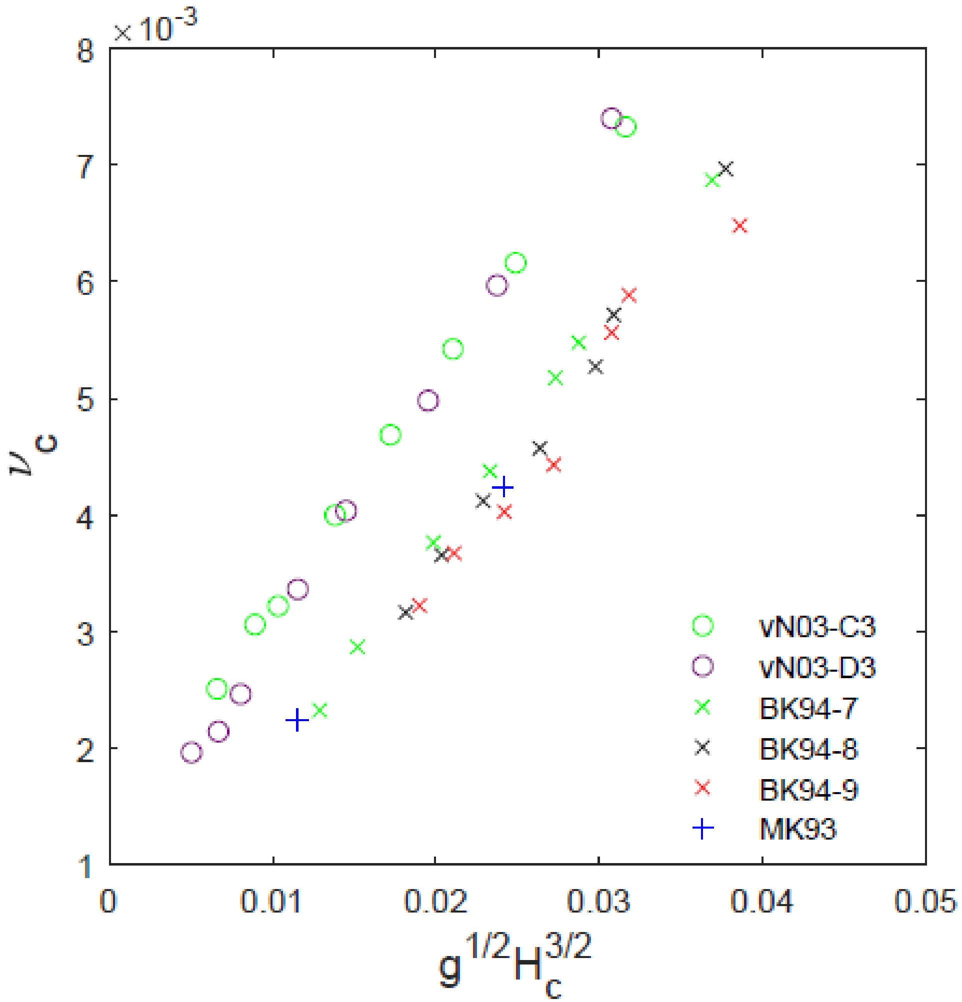
$$\frac{E}{E_a} = \text{csch}^2\left(\frac{\omega}{\omega_\nu}\right)$$

$$E_a = \frac{8\nu_c^2\omega_m}{9g}$$



Assumption : ν_c is mainly controlled by H_c

$$\longrightarrow \quad \nu_c = \alpha_\nu g^{1/2} H_c^{3/2}$$



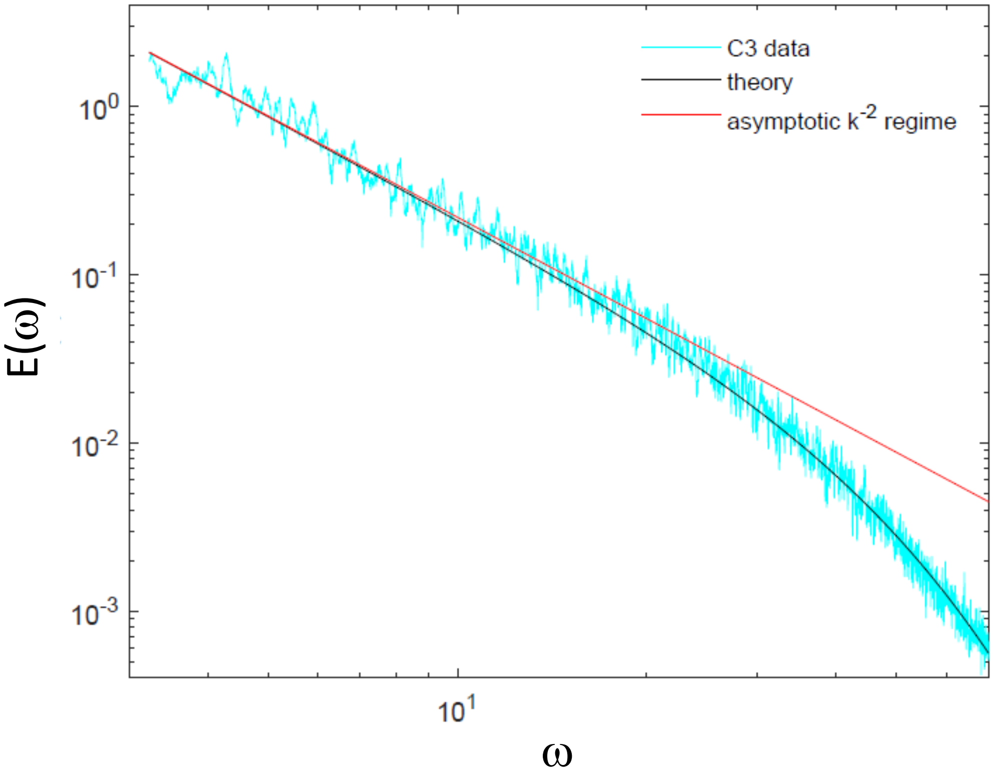
$$R_B = \frac{3}{2}\frac{gH_c T_m}{\nu_c}$$

$$R_B \sim H_c^{-1/2} T_m$$

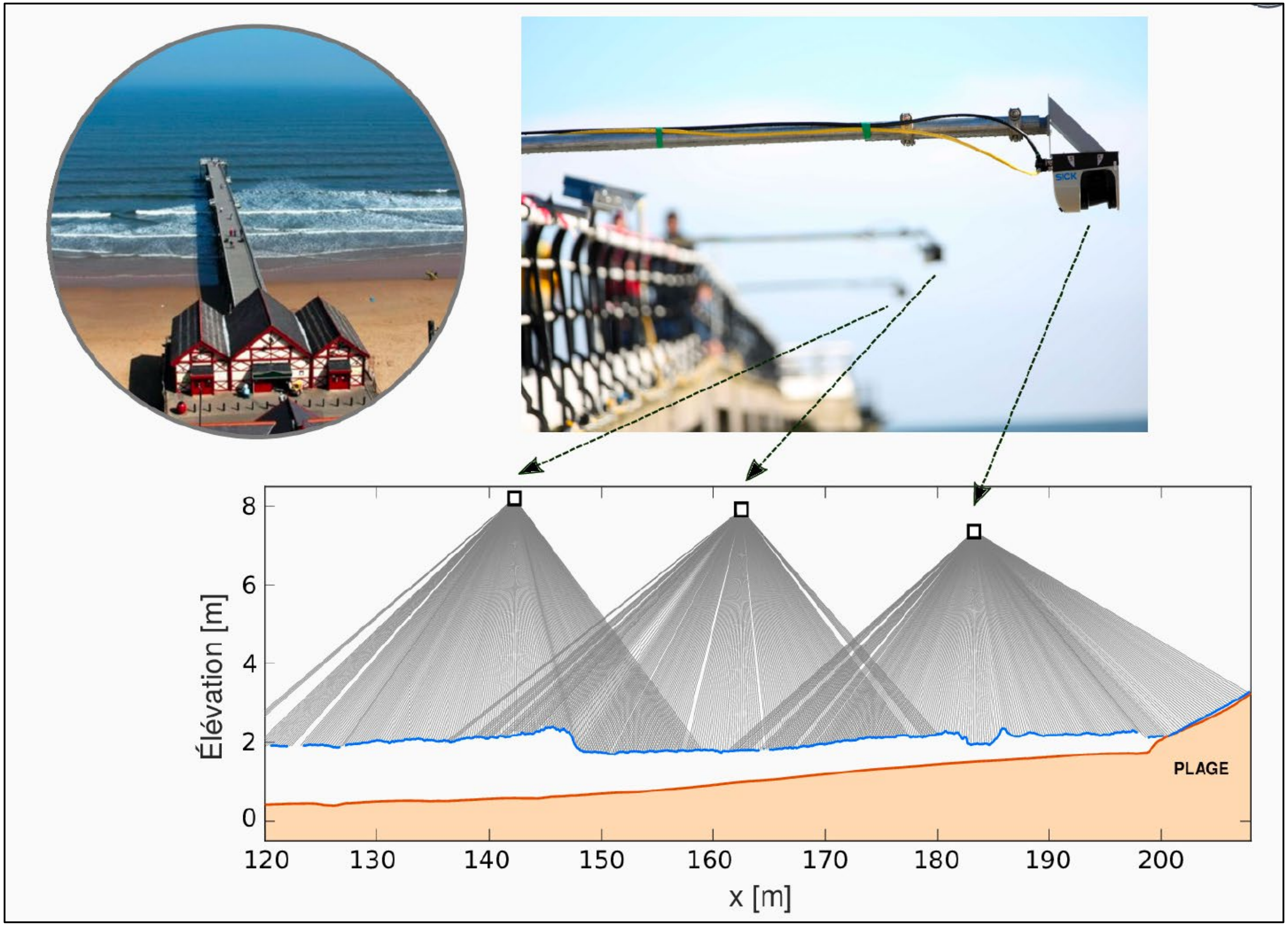
knowing ω_m and E_T

$$\nu_c = \alpha_\nu g^{1/2} H_c^{3/2}$$

$$E(\omega) = \frac{8}{9}\frac{\nu_c^2}{g}\omega_m \mathrm{csch}^2\left(\frac{\omega}{\omega_\nu}\right)$$

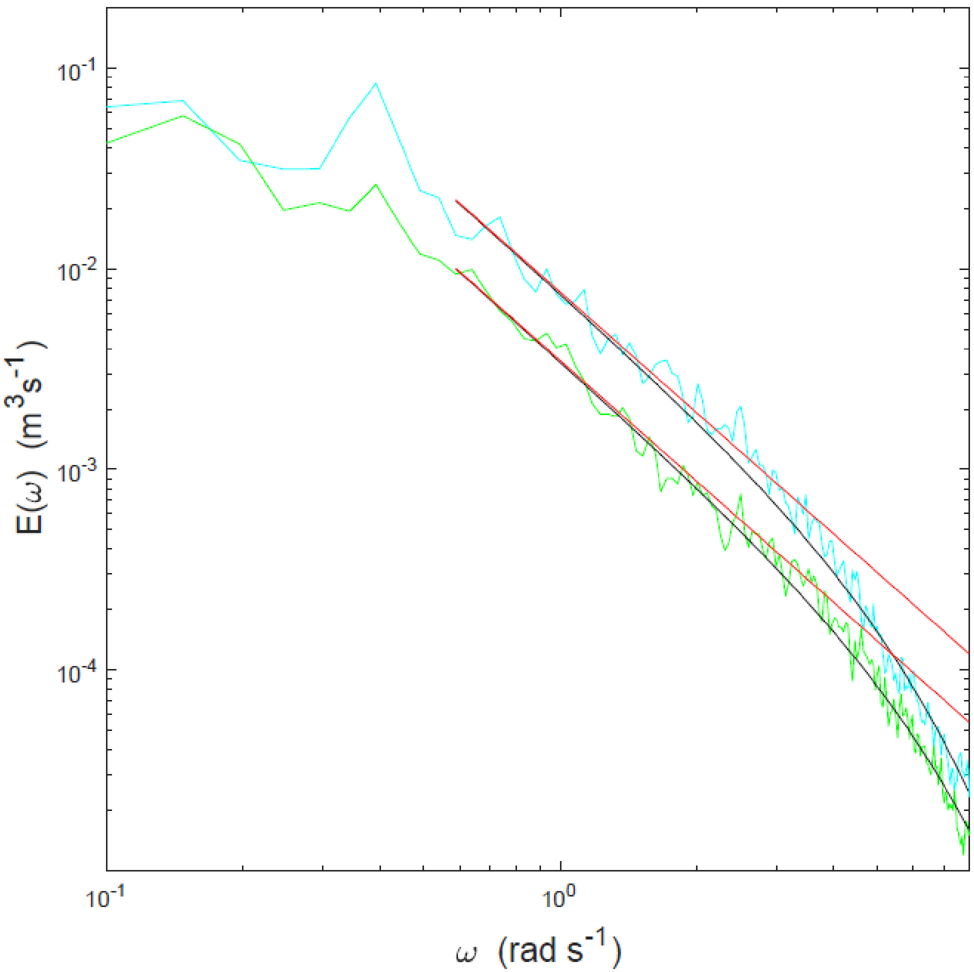


Kevin Martins (LIENSs, CNRS)



field campaign at Duck, North Carolina

(*Martins, Bonneton et al. Coastal Eng. 2025)*

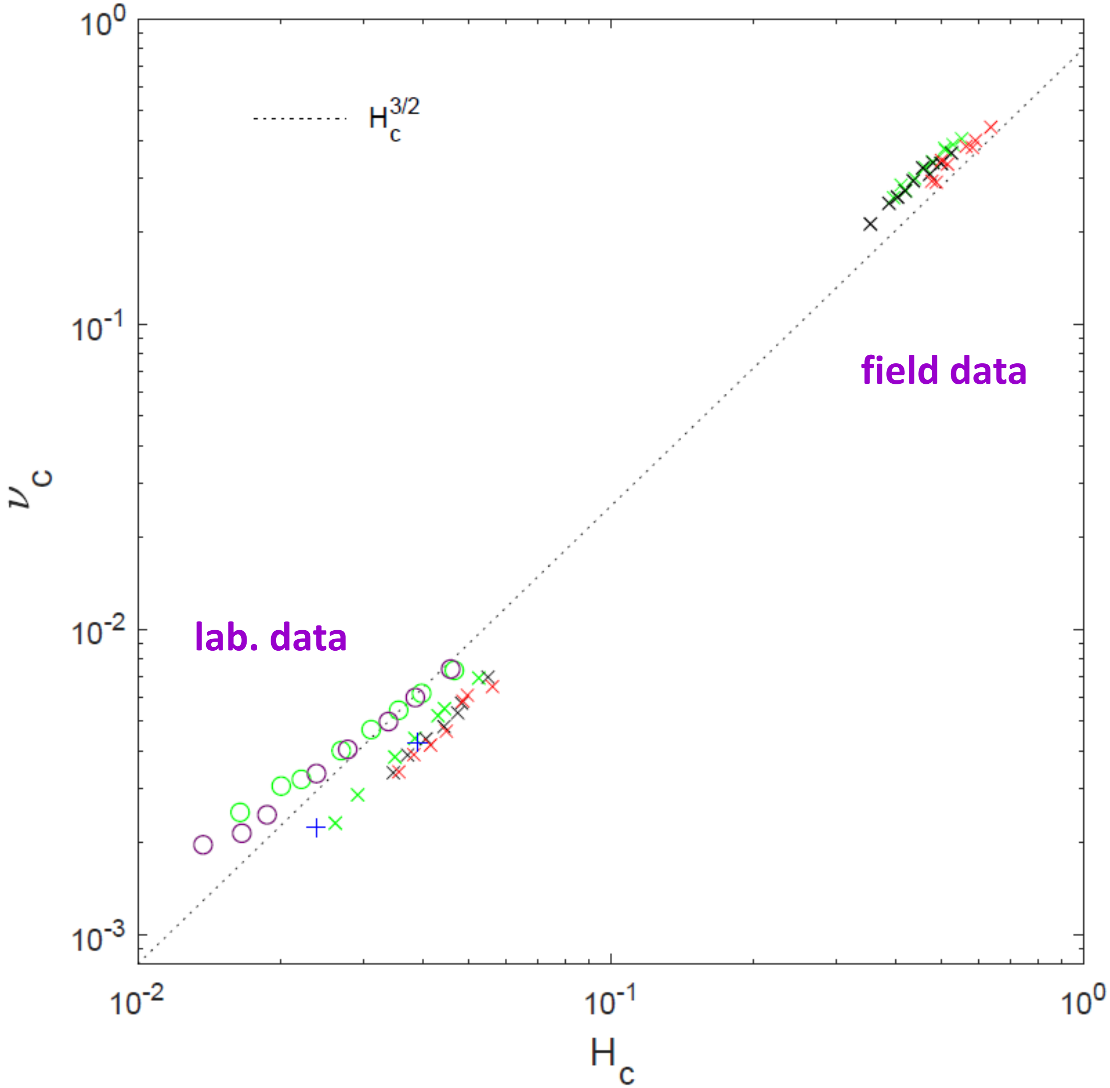


$$E(\omega) = \frac{8}{9}\frac{\nu_c^2}{g}\omega_m \mathrm{csch}^2\left(\frac{\omega}{\omega_\nu}\right)$$

cyan, H_s=0.71 m, ω_ν= 3.4 rad/s
vert, H_s= 0.51 m, ω_ν= 3.9 rad/s

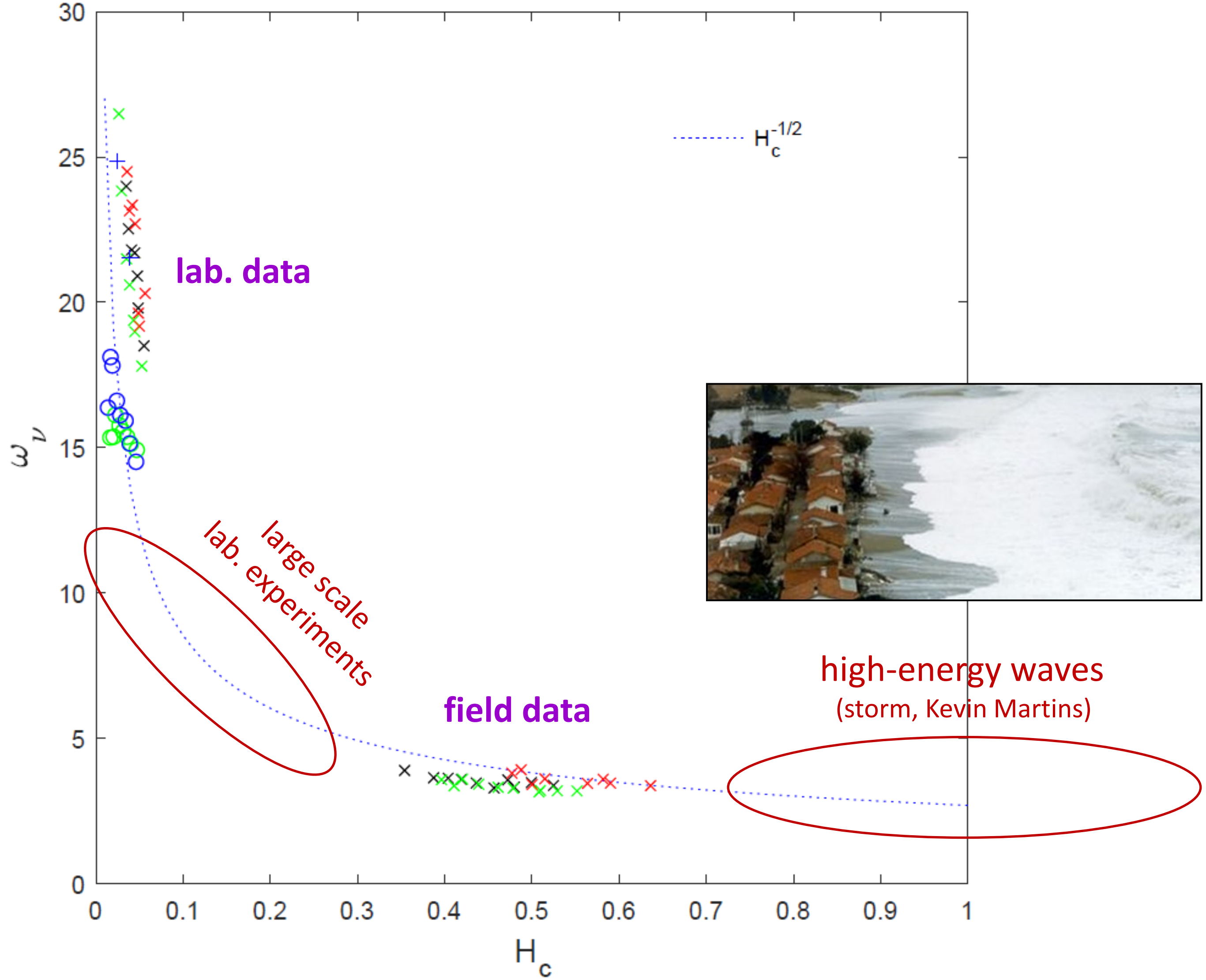
Field measurements – preliminary results

$$\nu_c = \alpha_\nu g^{1/2} H_c^{3/2}$$



Field measurements – preliminary results

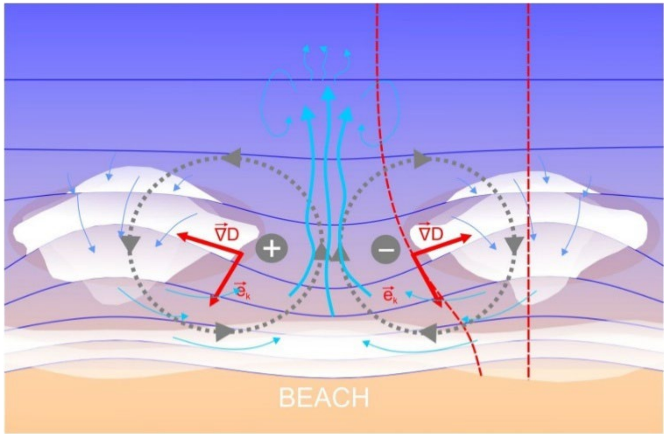
$$\nu_c = \alpha_\nu g^{1/2} H_c^{3/2} \longrightarrow \omega_\nu = \frac{3}{4\pi\alpha_\nu} g^{1/2} H_c^{-1/2}$$



Energy dissipation spectrum

Energy dissipation spectrum

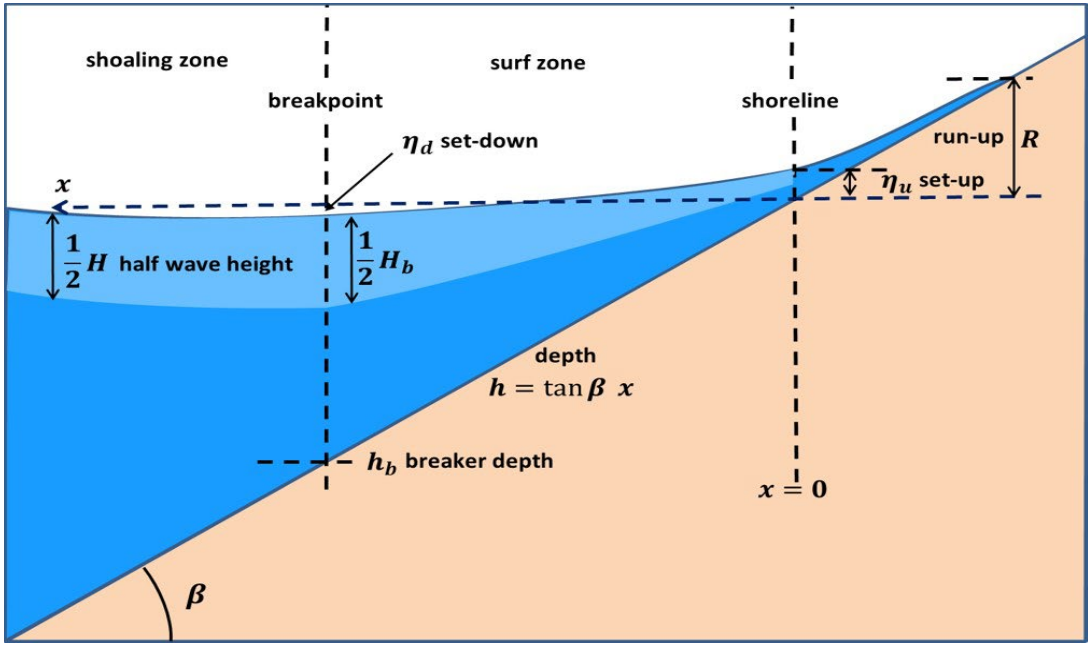
- wave energy dissipation: one of the most important processes in the surf zone
- dissipation and its spatial variation control the mean wave-induced circulation



Castelle et al. 2013

Energy dissipation spectrum

- wave energy dissipation: one of the most important processes in the surf zone
- dissipation and its spatial variation control the mean wave-induced circulation
- wave set-up



Energy dissipation spectrum

- wave energy dissipation: one of the most important processes in the surf zone
- dissipation and its spatial variation control the mean wave-induced circulation
- wave set-up
- crucial to have a good parametrization of D(ω) in spectral wave models

$$\frac{\partial E(\omega, x, t)}{\partial t} + \frac{\partial}{\partial x}\left(c_0 E(\omega, x, t)\right) = -T(\omega, x, t) - D(\omega, x, t)$$

Energy dissipation spectrum

$$D(k) = 2\nu_c k^2 E(k)$$

$$D(\omega) = \frac{2\nu_c}{gh_0}\omega^2 E(\omega)$$

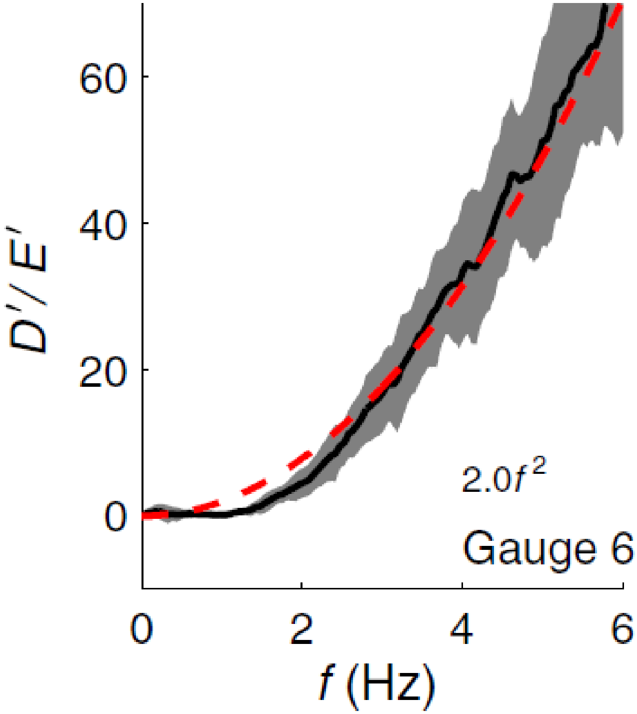
$$\frac{D(\omega)}{E(\omega)} = \frac{2\nu_c}{gh_0}\omega^2$$

operational spectral models

WWIII, SWAN, TOMAWAC, …

$$\frac{D(\omega)}{E(\omega)} = \frac{\boldsymbol{D}}{\boldsymbol{E}}$$

Eldeberky and Battjes 1996



Kirby and Kaihatu 1997

Smit et al. 2014

Energy dissipation spectrum

$$D(k) = 2\nu_c k^2 E(k)$$

$$D(\omega) = \frac{2\nu_c}{gh_0}\omega^2 E(\omega)$$

$$\frac{D(\omega)}{E(\omega)} = \frac{2\nu_c}{gh_0}\omega^2$$

operational spectral models

WWIII, SWAN, TOMAWAC, …

$$\frac{D(\omega)}{E(\omega)} = \frac{\boldsymbol{D}}{\boldsymbol{E}}$$

Eldeberky and Battjes 1996

↓

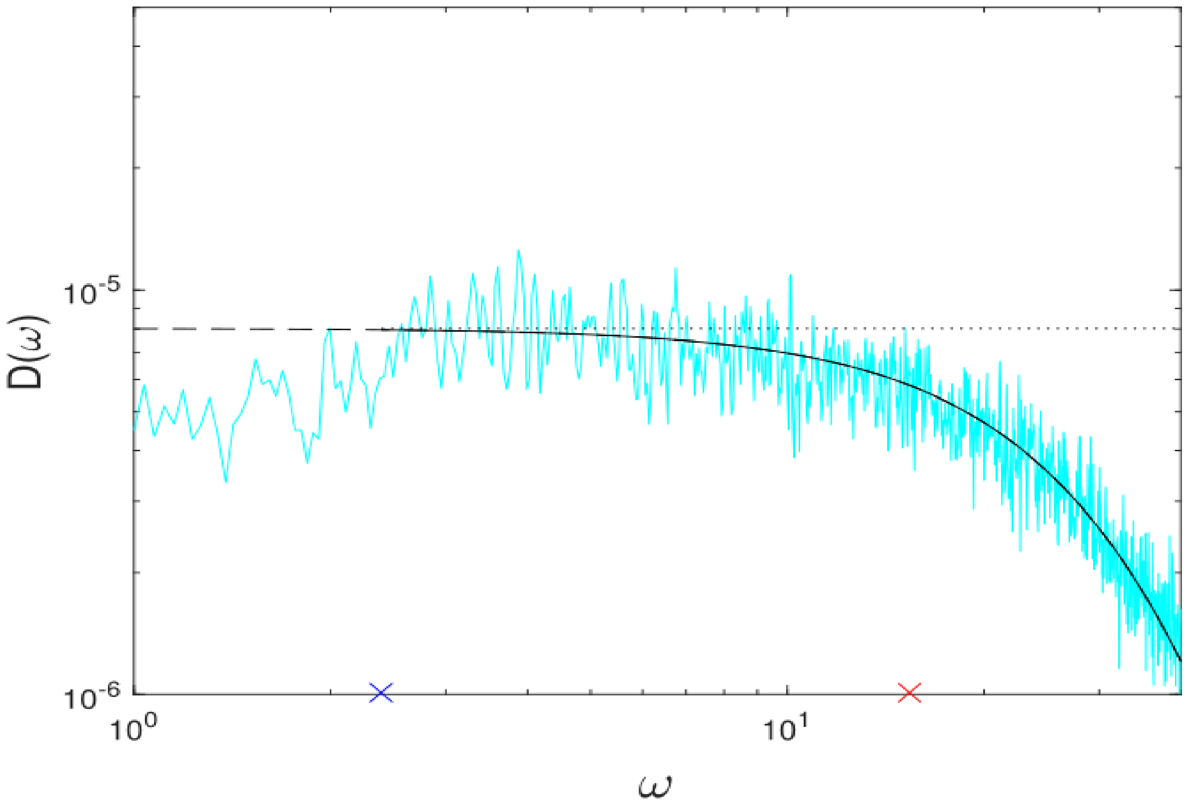
does not depend on any empirical estimation of D

Battjes and Janssen 1978 or Thornton & Guza 1983

Energy dissipation spectrum

$$D(\omega) = \frac{16}{9}\frac{\nu_c^3}{g^2 h_0}\omega_m \omega^2 \text{csch}^2\left(\frac{\omega}{\omega_\nu}\right)$$

$$\omega \ll \omega_\nu \qquad D(\omega) = \frac{\nu_c \omega_m H_c^2}{\pi^2 h_0}$$



Conclusion

our work is based on the hypothesis that:

in the ISZ, the shape of E(ω) is similar to the one given

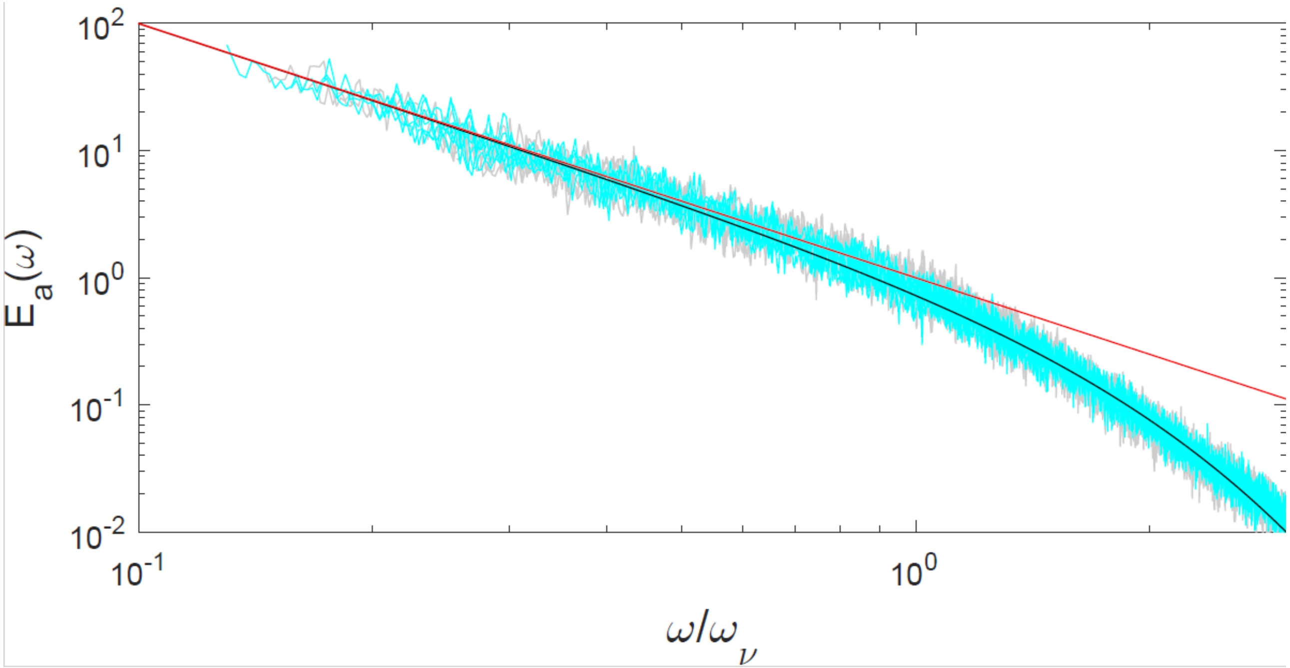
by the Burgers-type nonlinear shallow water model

$$\frac{\partial \zeta}{\partial t} + c_0 \frac{\partial \zeta}{\partial x} + \frac{3}{2} c_0 \frac{\zeta}{h_0} \frac{\partial \zeta}{\partial x} = \nu_t \frac{\partial^2 \zeta}{\partial x^2}$$

somewhat exploratory $\rightarrow$ convincing results

Conclusion

- the elevation spectrum has an universal shape



- Saffman-type equation

$$\frac{E}{E_a} = \operatorname{csch}^2\left(\frac{\omega}{\omega_\nu}\right)$$

$$E_a = \frac{8\nu_c^2\omega_m}{9g}$$

Conclusion

- in the ISZ, the dissipation spectrum is related to $E(\omega)$ by

$$D(\omega) = \frac{2\nu_c}{g h_0}\omega^2 E(\omega)$$

$\rightarrow$ could inspire new parametrizations in operational spectral wave models for the **entire surf zone**

- this work could also serve as a basis for characterizing the diffusive regime, $\omega > \omega_\nu$, in phase-resolving wave models (Boussinesq, SGN, …)

Thank you for your attention