The GroundWater Waves problem in the Dupuit-Forchheimer regime

Martin Parisot - Cardamom Inria Bordeaux
in collaboration with Manon Carreau

Inria

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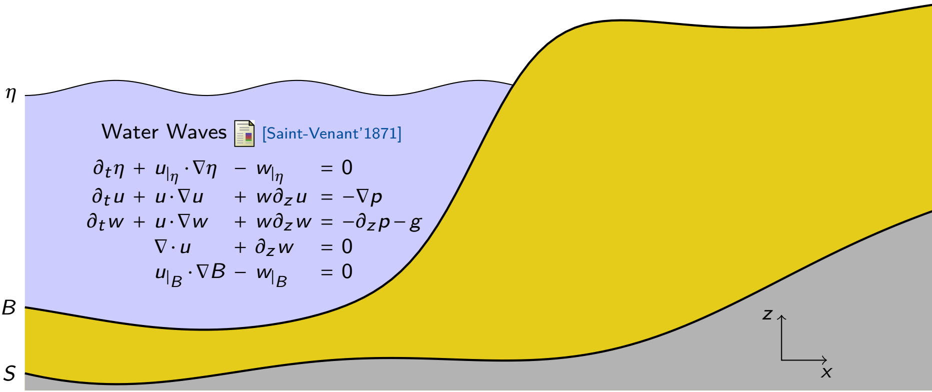
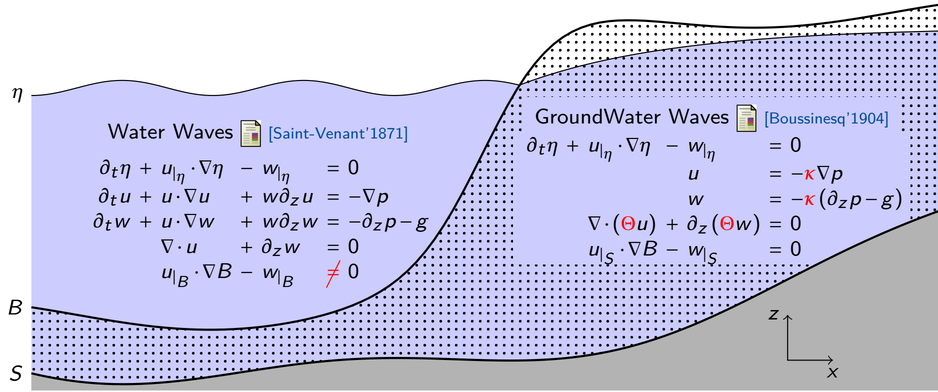


Water Waves [Saint-Venant'1871]

$$
\begin{aligned}
\partial_t \eta + u_{|\eta} \cdot \nabla \eta - w_{|\eta} &= 0 \\
\partial_t u + u \cdot \nabla u + w \partial_z u &= -\nabla p \\
\partial_t w + u \cdot \nabla w + w \partial_z w &= -\partial_z p - g \\
\nabla \cdot u + \partial_z w &= 0 \\
u_{|B} \cdot \nabla B - w_{|B} &= 0
\end{aligned}
$$



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\partial_t u + u\cdot\nabla u + w\partial_z u &= -\nabla p\\
\partial_t w + u\cdot\nabla w + w\partial_z w &= -\partial_z p - g\\
\nabla\cdot u + \partial_z w &= 0\\
u_{|B}\cdot\nabla B - w_{|B} &\neq 0
\end{aligned}$$

GroundWater Waves [Boussinesq'1904]

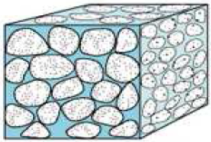
$$\begin{aligned}
\partial_t \eta + u_{|\eta}\cdot\nabla\eta - w_{|\eta} &= 0\\
u &= -\kappa\nabla p\\
w &= -\kappa(\partial_z p - g)\\
\nabla\cdot(\Theta u) + \partial_z(\Theta w) &= 0\\
u_{|S}\cdot\nabla B - w_{|S} &= 0
\end{aligned}$$

$\Theta(x,z)$: porosity $= \dfrac{\text{void-space volume}}{\text{total volume}} \leq 1$

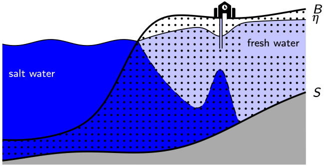
$\kappa(x,z)$: permeability (or hydraulic conductivity)

$B(x)$: bathymetry

$S(x)$: substratum



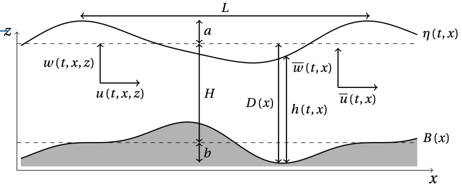
- Water Resources: baseflow, flooding, salt intrusion, pollutant...



- Geomorphology: sediment transport, soil alteration...

GroundWater Waves model (*GWW*):

$$\begin{aligned}\partial_t\eta + u_{|\eta}\cdot\nabla\eta - w_{|\eta} &= 0\\ u &= -\kappa\nabla p\\ w &= -\kappa(\partial_z p + g)\\ \nabla\cdot(\Theta u) + \partial_z(\Theta w) &= 0\\ u_{|S}\cdot\nabla S - w_{|S} &= 0\end{aligned}$$



Properties

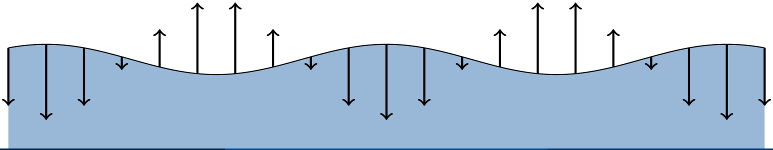
- **Energy dissipation**: $\partial_t\mathscr{P}(\eta) + \nabla\cdot\mathscr{G}(\eta,u,w) = -\widetilde{\mathscr{K}}(\eta,u,w)$

  where $\mathscr{P}(\eta) := g\int_S^\eta \Theta z\,\mathrm{d}z$ (potential energy)

  $\mathscr{G}(\eta,u,w) := \int_S^\eta (gz+p)\Theta u\,\mathrm{d}z$ (energy flux)

  and $\widetilde{\mathscr{K}}(\eta,u,w) := \int_S^\eta \frac{\Theta}{\kappa}\left(|u|^2+w^2\right)\mathrm{d}z.$ (dissipation ≈ kinetic energy)

- The **linearized** groundwater waves model with constant parameters has solutions of the form $\eta(t,x) = D\cos(k\cdot x)e^{-\frac{g\kappa}{D}\widetilde{\lambda}t}$ with the **linear decay rate** reads $\widetilde{\lambda} = |kD|\tanh(|kD|)$.



- Integrating the incompressibility constraint, we get $\partial_t \mathcal{V}(\eta) + \nabla\cdot(\mathcal{V}(\eta)\overline{u}) = 0$
with the water volume $\mathcal{V}(\eta) := \int_S^\eta \Theta\,\mathrm{d}z$, and the mean horizontal velocity $\overline{u} := \frac{1}{\mathcal{V}(\eta)}\int_S^\eta \Theta u\,\mathrm{d}z$.

- Integrating the horizontal Darcy law, we get $\mathcal{V}(\eta)\overline{u} = -\int_S^\eta \kappa\nabla p\,\mathrm{d}z$

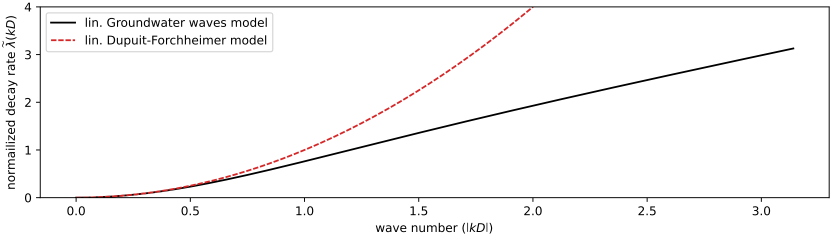
Dupuit–Forchheimer assumption

$$p(t,x,z) = g(\eta(t,x) - z) + O(\mu) \quad \Rightarrow \quad \overline{u} = -gK(\eta)\nabla\eta + O(\mu) \quad \text{with } K(\eta) = \frac{1}{\mathcal{V}(\eta)}\int_S^\eta \Theta\kappa\,\mathrm{d}z.$$

Dupuit-Forchheimer model (*DF*): $O(\mu)$-approximate model [Dupuit'1863]

$$\partial_t \mathcal{V}(\eta) - \nabla\cdot(gK(\eta)\mathcal{V}(\eta)\nabla\eta) = 0$$

- **Energy dissipation**: $\partial_t \mathscr{P}(\eta) + \nabla\cdot\mathscr{G}(\eta,u,w) = -\widetilde{\mathscr{K}}(\eta,u,w)$
where $\mathscr{P}(\eta) := g\int_S^\eta \Theta z\,\mathrm{d}z$, $\mathscr{G}(\eta,u,w) := g^2\eta K\mathcal{V}\nabla\eta$ and $\widetilde{\mathscr{K}}(\eta,u,w) := g^2 K\mathcal{V}|\nabla\eta|^2$.
- **Linear decay rate**: $\widetilde{\lambda} = |kD|^2$.



- **Assuming** constant coefficients: $\Theta(x,z) = \overline{\Theta}(x) + O\left(\mu^2\right)$ and $\kappa(x,z) = \overline{\kappa}(x) + O\left(\mu^2\right)$
- **Assuming** the hydrodynamic pressure small: $p(t,x,z) - g(\eta(t,x) - z) = O(\mu) = \mu q(t,x,z)$

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- **Assuming** the hydrodynamic pressure small: $p(t,x,z) - g(\eta(t,x) - z) = O(\mu) = \mu q(t,x,z)$

- Form the horizontal Darcy law: $u = -\overline{\kappa}\nabla(g\eta + \mu q)$

  we get $h\overline{u} = -g\overline{\kappa}h\nabla(h+S) - \mu\overline{\kappa}\left(\nabla(h\overline{q}) + \beta q_S \nabla S\right) + O\left(\mu^2\right)$

  with $\overline{q}(t,x) = \frac{1}{h(t,x)}\int_S^{\eta(t,x)} q(t,x,z)\,\mathrm{d}z$ and $q_S(t,x) = q(t,x,S(x)).$

- **Assuming** constant coefficients: $\Theta(x,z) = \overline{\Theta}(x) + O\left(\mu^2\right)$ and $\kappa(x,z) = \overline{\kappa}(x) + O\left(\mu^2\right)$
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- Form the incompressibility constraint: $\nabla\cdot\left(\overline{\Theta}u\right) + \partial_z\left(\overline{\Theta}w\right) = 0$

we get $w(t,x,z) = \overline{w}(t,x) + \dfrac{z - S - \frac{h}{2}}{h}\widetilde{w}(t,x) + O\left(\mu^2\right)$ with $\overline{w} = \mu\left(\beta\overline{u}\cdot\nabla S - \frac{h}{2\overline{\Theta}}\nabla\cdot\left(\overline{\Theta}\,\overline{u}\right)\right)$

and $\widetilde{w} = -\frac{h}{\overline{\Theta}}\nabla\cdot\left(\overline{\Theta}\,\overline{u}\right)$

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- **Assuming** the hydrodynamic pressure small: $p(t,x,z) - g(\eta(t,x) - z) = O(\mu) = \mu q(t,x,z)$

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and $\widetilde{w} = -\frac{h}{\overline{\Theta}}\nabla\cdot\left(\overline{\Theta}\,\overline{u}\right)$

- Form the vertical Darcy law: $w = -\overline{\kappa}\partial_z q$

  we get $h\overline{w} = \overline{\kappa}q_S$ and $h\widetilde{w} = 6\overline{\kappa}\left(2\overline{q} - q_S\right)$

- **Assuming** constant coefficients: $\Theta(x,z) = \overline{\Theta}(x) + O\left(\mu^2\right)$ and $\kappa(x,z) = \overline{\kappa}(x) + O\left(\mu^2\right)$
- **Assuming** the hydrodynamic pressure small: $p(t,x,z) - g(\eta(t,x) - z) = O(\mu) = \mu q(t,x,z)$

- Form the horizontal Darcy law: $u = -\overline{\kappa}\nabla(g\eta + \mu q)$

  we get $h\overline{u} = -g\overline{\kappa}h\nabla(h+S) - \mu\overline{\kappa}\left(\nabla(h\overline{q}) + \beta q_S \nabla S\right) + O\left(\mu^2\right)$

  with $\overline{q}(t,x) = \frac{1}{h(t,x)}\int_S^{\eta(t,x)} q(t,x,z)\,\mathrm{d}z$ and $q_S(t,x) = q(t,x,S(x))$.

- Form the incompressibility constraint: $\nabla\cdot\left(\overline{\Theta}u\right) + \partial_z\left(\overline{\Theta}w\right) = 0$

we get $w(t,x,z) = \overline{w}(t,x) + \dfrac{z - S - \frac{h}{2}}{h}\widetilde{w}(t,x) + O\left(\mu^2\right)$ with $\overline{w} = \mu\left(\beta\overline{u}\cdot\nabla S - \frac{h}{2\overline{\Theta}}\nabla\cdot\left(\overline{\Theta}\overline{u}\right)\right)$

and $\widetilde{w} = -\frac{h}{\overline{\Theta}}\nabla\cdot\left(\overline{\Theta}\overline{u}\right)$

- Form the vertical Darcy law: $w = -\overline{\kappa}\partial_z q$

  we get $h\overline{w} = \overline{\kappa}q_S$ and $h\widetilde{w} = 6\overline{\kappa}\left(2\overline{q} - q_S\right)$

- We conclude the equation on the horizontal velocity: $\left(\mathrm{I_d} + \mu^2\mathscr{T}_{S,h}^{\overline{\kappa},\overline{\Theta}}\right)\overline{u} = -g\overline{\kappa}\nabla(h+S)$

  with $\mathscr{T}_{\mathbf{b},\mathbf{h}}^{\overline{\kappa},\overline{\Theta}}(V) = \beta^2\nabla\mathbf{b}\otimes\nabla\mathbf{b}V + \frac{\overline{\kappa}}{\mathbf{h}}\nabla\left(\frac{\beta\mathbf{h}^2\nabla\mathbf{b}}{2\overline{\kappa}}\cdot V\right) - \frac{\beta\mathbf{h}^2\nabla\mathbf{b}}{2\overline{\Theta}\mathbf{h}}\nabla\cdot\left(\overline{\Theta}V\right) + \frac{\overline{\kappa}}{\mathbf{h}}\nabla\left(\frac{\mathbf{h}^3}{3\overline{\kappa}\overline{\Theta}}\nabla\cdot\left(\overline{\Theta}V\right)\right)$.

[Lannes'13]

Hydrodynamic Dupuit-Forchheimer model ($HDF_{S,h}$): $O\left(\mu^2\right)$-approximate model

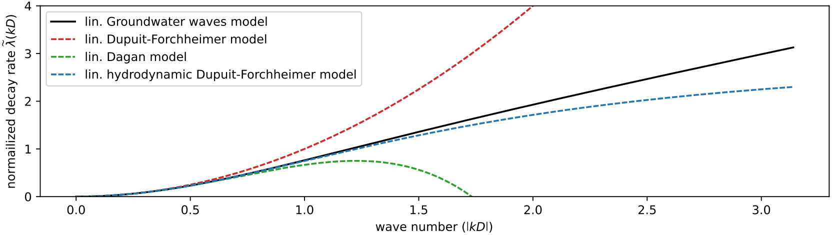
$$\partial_t\left(\overline{\Theta}h\right) - \nabla\cdot\left(\overline{\Theta}h\left(\mathrm{I_d} + \mu^2\mathcal{T}_{S,h}^{\overline{\kappa},\overline{\Theta}}\right)^{-1}\left(g\overline{\kappa}\nabla\left(h+S\right)\right)\right) = 0$$

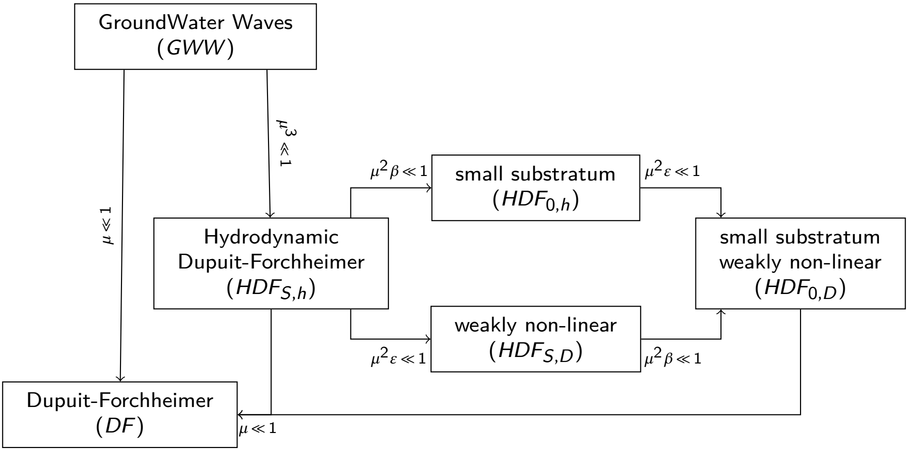
[Bucur'16]

- **Energy dissipation**: $\partial_t\mathscr{P}\left(h\right) + \nabla\cdot\mathscr{G}\left(h,u,w\right) = -\widetilde{\mathscr{K}}\left(h,u,w\right)$

where
$$\mathscr{P}\left(h\right) := g\overline{\Theta}h\left(S+\frac{h}{2}\right)$$
$$\mathscr{G}\left(h,u,w\right) := \left(g\left(h+S\right)+\mu q\right)\overline{\Theta}h\overline{u}$$
and
$$\widetilde{\mathscr{K}}\left(h,u,w\right) := \frac{\overline{\Theta}h}{\overline{\kappa}}\left(\left|\overline{u}\right|^2 + \overline{w}^2 + \frac{\widetilde{w}^2}{12}\right).$$

- **Linear decay rate**: $\widetilde{\lambda} = \frac{|kD|^2}{1+\frac{|kD|^2}{3}} \underset{|kD|\to\infty}{\longrightarrow} 3.$




GroundWater Waves (GWW)
Hydrodynamic Dupuit-Forchheimer ($HDF_{S,h}$)
small substratum ($HDF_{0,h}$)
weakly non-linear ($HDF_{S,D}$)
small substratum weakly non-linear ($HDF_{0,D}$)
Dupuit-Forchheimer (DF)
$\mu \ll 1$
$\mu^3 \ll 1$
$\mu^2 \beta \ll 1$
$\mu^2 \varepsilon \ll 1$
$\mu^2 \varepsilon \ll 1$
$\mu^2 \beta \ll 1$
$\mu \ll 1$

Numerical scheme:

$$h_k^{n+1} = h_k^n - \frac{\delta_t}{\overline{\Theta}_k} \nabla_k^\delta \cdot \left(\overline{\Theta}_\star \left\{ h_\star^n \right\}_\star \overline{u}_\star^{n+1} \right)$$

$$\left(1 + \left[\mathscr{T}_{S_\star, h_\star^n}^{\overline{\kappa}, \overline{\Theta}} \right]_f \right) \left(\overline{u}_\star^{n+1} \right) = -g \overline{\kappa}_f \nabla_f^\delta \left(h_\star^{n+1} + S_\star \right).$$

The discrete space differential operator is defined as

$$\left[\mathscr{T}_{\mathbf{b}_\star, \mathbf{h}_\star}^{\overline{\kappa}, \overline{\Theta}} \right]_f (U_\star) = \frac{\overline{\kappa}_f}{\{\mathbf{h}_\star\}_f} \left\{ \frac{\mathbf{h}_\star}{\overline{\kappa}_\star \overline{\Theta}_\star} \left\{ \overline{\Theta}_\star U_\star \cdot \nabla_\star^\delta \mathbf{b}_\star \right\}_\star \right\}_f \nabla_f^\delta \mathbf{b}_\star$$
$$+ \frac{\overline{\kappa}_f}{\{\mathbf{h}_\star\}_f} \left(\nabla_f^\delta \left(\frac{|\mathbf{h}_\star|^2}{2 \overline{\kappa}_\star \overline{\Theta}_\star} \left\{ \overline{\Theta}_\star U_\star \cdot \nabla_\star^\delta \mathbf{b}_\star \right\}_\star \right) - \left\{ \frac{|\mathbf{h}_\star|^2}{2 \overline{\kappa}_\star \overline{\Theta}_\star} \nabla_\star^\delta \cdot \left(\overline{\Theta}_\star U_\star \right) \right\}_f \nabla_f^\delta \mathbf{b}_\star \right)$$
$$- \frac{\overline{\kappa}_f}{\{\mathbf{h}_\star\}_f} \nabla_f^\delta \left(\frac{|\mathbf{h}_\star|^3}{3 \overline{\kappa}_\star \overline{\Theta}_\star} \nabla_\star^\delta \cdot \left(\overline{\Theta}_\star U_\star \right) \right).$$

The discrete divergence and gradient operators are defined as

$$\nabla_k^\delta \cdot U_\star = \frac{1}{\mathfrak{m}_k} \sum_{f \in \mathbb{F}_k} U_f \cdot \mathfrak{n}_k^{k_f} \mathfrak{m}_f \quad \text{and} \quad \nabla_f^\delta \phi_\star = \frac{\phi_{k_f} - \phi_k}{\delta_f} \mathfrak{n}_k^{k_f}$$

$$\{\psi_\star\}_k = \frac{1}{\# \mathbb{F}_k} \sum_{f \in \mathbb{F}_k} \psi_f \quad \text{and} \quad \{\phi_\star\}_f = \frac{\phi_{k_f} + \phi_k}{2}.$$

Entropy satisfying: The scheme satisfies a discrete counterpart of the energy dissipation.

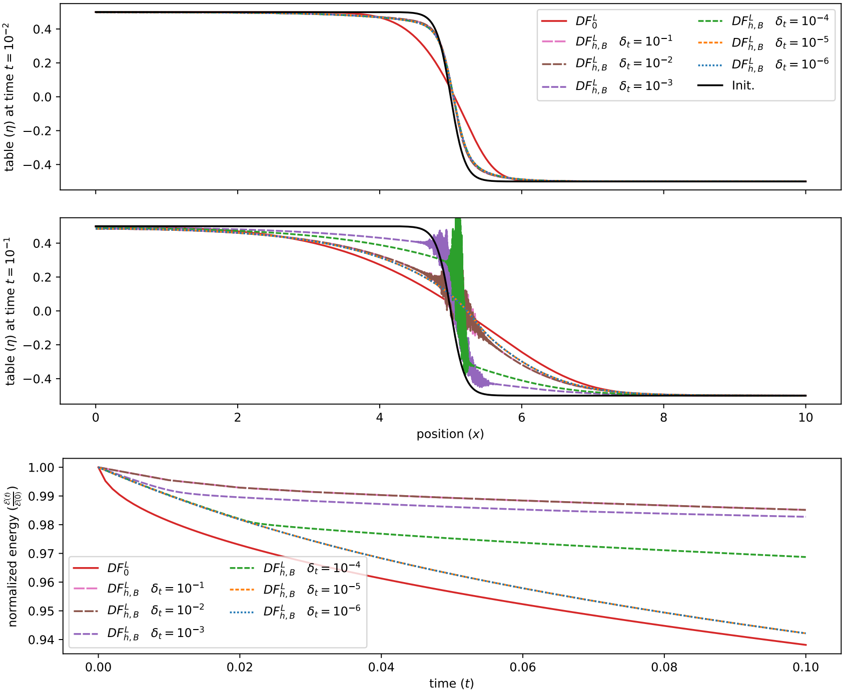
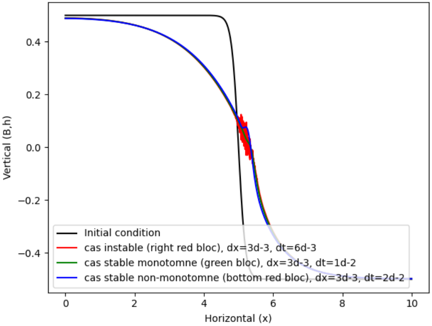
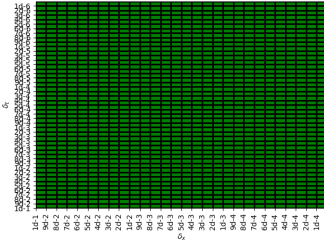
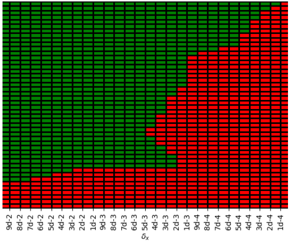


table (η) at time t = 10^−2
table (η) at time t = 10^−1
position (x)
normalized energy (ℰ(t)/ℰ(0))
time (t)
DF_0^L
DF_{h,B}^L δ_t = 10^−1
DF_{h,B}^L δ_t = 10^−2
DF_{h,B}^L δ_t = 10^−3
DF_{h,B}^L δ_t = 10^−4
DF_{h,B}^L δ_t = 10^−5
DF_{h,B}^L δ_t = 10^−6
Init.

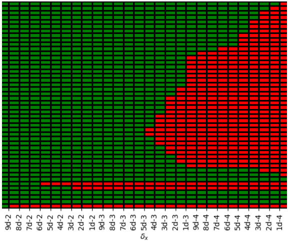




fully linear



weakly non-linear



fully non-linear

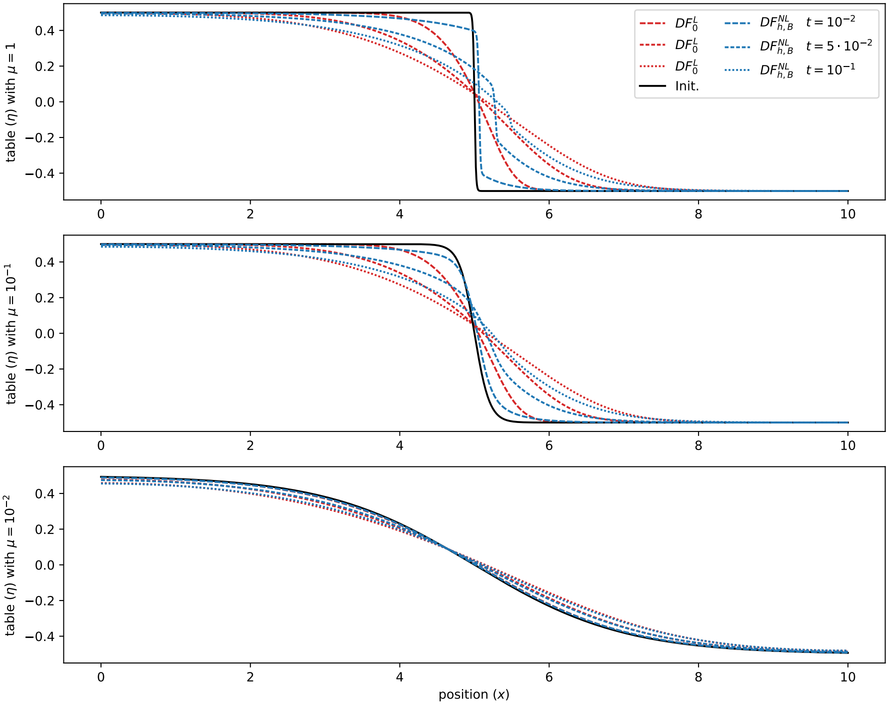


table (η) with $\mu = 1$
table (η) with $\mu = 10^{-1}$
table (η) with $\mu = 10^{-2}$
position (x)
DF_0^L
DF_0^L
DF_0^L
Init.
$DF_{h,B}^{NL}$ $t = 10^{-2}$
$DF_{h,B}^{NL}$ $t = 5 \cdot 10^{-2}$
$DF_{h,B}^{NL}$ $t = 10^{-1}$

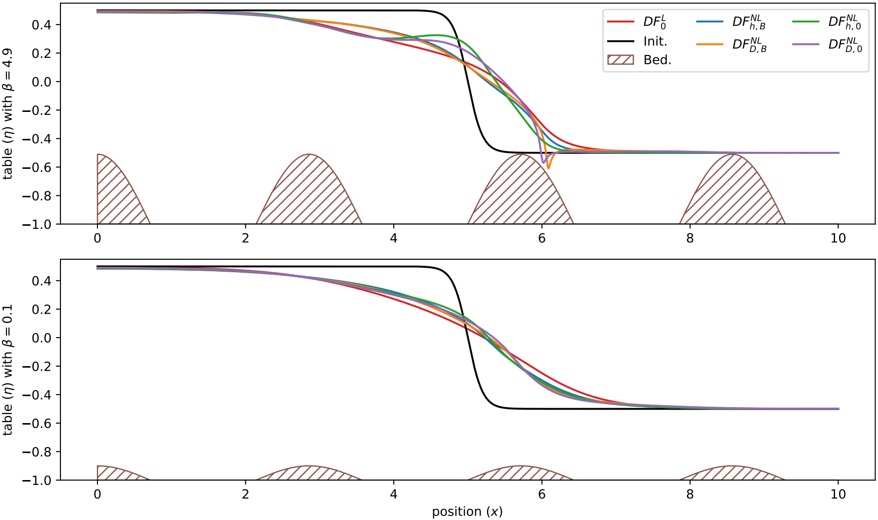
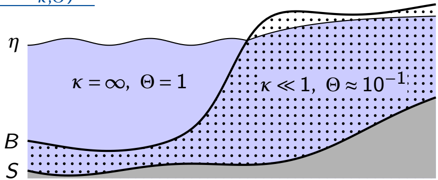


table (η) with β = 4.9
table (η) with β = 0.1
position (x)
DF_0^L
Init.
Bed.
DF_{h,B}^{NL}
DF_{D,B}^{NL}
DF_{h,0}^{NL}
DF_{D,0}^{NL}
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
0
2
4
6
8
10

Water Waves model with drag force ($WW_{\kappa,\Theta}$):

$$\begin{aligned}
\partial_t \eta + u_{|\eta} \cdot \nabla \eta - w_{|\eta} &= 0 \\
\partial_t u + u \cdot \nabla u + w \partial_z u &= -\nabla p - \frac{u}{\kappa} \\
\partial_t w + u \cdot \nabla w + w \partial_z w &= -\partial_z p - g - \frac{w}{\kappa} \\
\nabla \cdot (\Theta u) + \partial_z (\Theta w) &= 0 \\
u_{|S} \cdot \nabla S - w_{|S} &= 0
\end{aligned}$$



Properties

- **Asymptotic regime**:

$$(WW) \xleftarrow[\Theta=1]{\kappa\to\infty} (WW_{\kappa,\Theta}) \xrightarrow{\kappa\to 0} (GWW)$$

$$(WW) \xrightarrow{\mu\to 0} (SW), \qquad (GWW) \xrightarrow{\mu\to 0} (DF)$$

- **Energy dissipation**: $\partial_t \left(\mathscr{P}(\eta) + \int_S^\eta \Theta \mathscr{K}(u,w)\,\mathrm{d}z \right) + \nabla \cdot \mathscr{G}(\eta,u,w) = -\widetilde{\mathscr{K}}(\eta,u,w)$

where

$$\begin{aligned}
\mathscr{P}(\eta) &:= g \int_S^\eta \Theta z \,\mathrm{d}z \\
\mathscr{K}(u,w) &:= \frac{1}{2}\left(|u|^2 + w^2\right) \\
\mathscr{G}(\eta,u,w) &:= \int_S^\eta (gz + \mathscr{K}(u,w) + p)\Theta u \,\mathrm{d}z
\end{aligned}$$

and $\widetilde{\mathscr{K}}(\eta,u,w) := \int_S^\eta \frac{\Theta}{\kappa}\left(|u|^2 + w^2\right)\mathrm{d}z.$

- Assuming **hydrostatic pressure** and **vertical-homogeneous horizontal velocity**.
- By vertical integration on $[S, \min(B,\eta)]$ and $[\min(B,\eta), \eta]$.

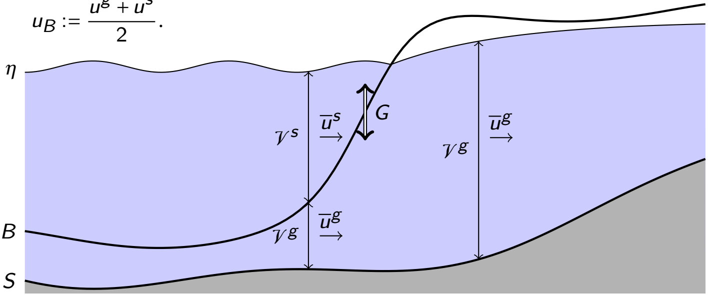
Bi-layer shallow water model $(SW^2_{\kappa,\Theta})$: [Rambaud'11]

$$
\begin{aligned}
\partial_t \mathcal{V}^s(\eta) + \nabla\cdot\left(\mathcal{V}^s(\eta)\overline{u}^s\right) &= -G\\
\partial_t \left(\mathcal{V}^s(\eta)\overline{u}^s\right) + \nabla\cdot\left(\mathcal{V}^s(\eta)\overline{u}^s\otimes\overline{u}^s\right) &= -g\mathcal{V}^s(\eta)\left(\nabla\eta + \frac{\overline{u}^s}{\overline{\kappa}^s}\right) - u_B G\\
\partial_t \mathcal{V}^g(\eta) + \nabla\cdot\left(\mathcal{V}^g(\eta)\overline{u}^g\right) &= G\\
\partial_t \left(\mathcal{V}^g(\eta)\overline{u}^g\right) + \nabla\cdot\left(\mathcal{V}^g(\eta)\overline{u}^g\otimes\overline{u}^g\right) &= -g\mathcal{V}^g(\eta)\left(\nabla\eta + \frac{\overline{u}^g}{\overline{\kappa}^g}\right) + u_B G
\end{aligned}
$$

with $\mathcal{V}^s(\eta) := \mathcal{V}(\eta) - \min\left(\mathcal{V}(\eta), \overline{V}\right)$, $\mathcal{V}^g(\eta) := \min\left(\mathcal{V}(\eta), \overline{V}\right)$, $\overline{V} = \mathcal{V}(B)$

$\overline{u}^s := \dfrac{1}{\mathcal{V}^s(\eta)}\displaystyle\int_{\min(B,\eta)}^{\eta} \Theta u\, dz$, $\overline{u}^g := \dfrac{1}{\mathcal{V}^g(\eta)}\displaystyle\int_{S}^{\min(B,\eta)} \Theta u\, dz$

and $u_B := \dfrac{\overline{u}^g + \overline{u}^s}{2}$.



- Assuming **hydrostatic pressure** and **vertical-homogeneous horizontal velocity**.
- By vertical integration on $[S,\min(B,\eta)]$ and $[\min(B,\eta),\eta]$.

Bi-layer shallow water model $(SW^2_{\kappa,\Theta})$:

[Rambaud'11]

$$
\begin{aligned}
\partial_t \mathcal{V}^s(\eta) + \nabla\cdot(\mathcal{V}^s(\eta)\overline{u}^s) &= -G \\
\partial_t (\mathcal{V}^s(\eta)\overline{u}^s) + \nabla\cdot(\mathcal{V}^s(\eta)\overline{u}^s \otimes \overline{u}^s) &= -g\mathcal{V}^s(\eta)\left(\nabla\eta + \frac{\overline{u}^s}{\overline{\kappa}^s}\right) - u_B G \\
\partial_t \mathcal{V}^g(\eta) + \nabla\cdot(\mathcal{V}^g(\eta)\overline{u}^g) &= G \\
\partial_t (\mathcal{V}^g(\eta)\overline{u}^g) + \nabla\cdot(\mathcal{V}^g(\eta)\overline{u}^g \otimes \overline{u}^g) &= -g\mathcal{V}^g(\eta)\left(\nabla\eta + \frac{\overline{u}^g}{\overline{\kappa}^g}\right) + u_B G
\end{aligned}
$$

with $\mathcal{V}^s(\eta) := \mathcal{V}(\eta) - \min\left(\mathcal{V}(\eta), \overline{V}\right)$, $\mathcal{V}^g(\eta) := \min\left(\mathcal{V}(\eta), \overline{V}\right)$, $\overline{V} = \mathcal{V}(B)$

$\overline{u}^s := \dfrac{1}{\mathcal{V}^s(\eta)} \displaystyle\int_{\min(B,\eta)}^{\eta} \Theta u \, dz$, $\overline{u}^g := \dfrac{1}{\mathcal{V}^g(\eta)} \displaystyle\int_{S}^{\min(B,\eta)} \Theta u \, dz$

and $u_B := \dfrac{\overline{u}^g + \overline{u}^s}{2}$.

Properties

- **Asymptotic regime**:

$$
\begin{array}{ccccc}
(WW) & \xleftarrow[\Theta=1]{\kappa\to\infty} & (WW_{\kappa,\Theta}) & \xrightarrow{\kappa\to 0} & (GWW) \\
\downarrow{\scriptstyle \mu\to 0} & & \downarrow{\scriptstyle \mu\to 0} & & \downarrow{\scriptstyle \mu\to 0} \\
(SW) & \xleftarrow[\Theta=1]{\kappa\to\infty} & (SW_{\kappa,\Theta}) & \xrightarrow{\kappa\to 0} & (DF)
\end{array}
$$

Unified shallow-water/Dupuit-Forchheimer model (SW/DF):

$$\begin{aligned} \partial_t \mathcal{V}(\eta) + \nabla\cdot\left(\mathcal{H}^s(\eta)\overline{u}^s\right) &= \nabla\cdot\left(g\mathcal{V}^g(\eta)\overline{\kappa}^g(\eta)\nabla\eta\right) \\ \partial_t\left(\mathcal{H}^s(\eta)\overline{u}^s\right) + \nabla\cdot\left(\mathcal{H}^s(\eta)\overline{u}^s\otimes\overline{u}^s\right) &= -g\mathcal{H}^s(\eta)\nabla\eta + \frac{\partial_t\mathcal{V}^s(\eta)+\nabla\cdot\left(\mathcal{V}^s(\eta)\overline{u}^s\right)}{2}\overline{u}^s \end{aligned}$$

with $\mathcal{H}^s(\eta) := \max(0, \eta - B)$

Properties

- **Energy dissipation**: $\partial_t\left(\mathcal{P}(\eta)+\mathcal{K}(\eta,\overline{u}^s)\right)+\nabla\cdot\mathcal{G}(\eta,\overline{u}^s) = -\widetilde{\mathcal{K}}(\eta)$

  where $\mathcal{P}(\eta) := g\int_S^\eta \Theta z\,\mathrm{d}z$, $\mathcal{K}(\eta,\overline{u}^s) := \frac{\mathcal{H}^s(\eta)}{2}|\overline{u}^s|^2$

  $$\mathcal{G}(\eta,\overline{u}^s) := \left(g\eta\mathcal{H}^s(\eta)+\mathcal{K}(\eta,\overline{u}^s)\right)\overline{u}^s - g^2\eta\mathcal{V}^g(\eta)\overline{\kappa}^g\nabla\eta$$

  and $\widetilde{\mathcal{K}}(\eta) := g^2\mathcal{V}^g(\eta)\overline{\kappa}^g\left|\nabla\eta\right|^2.$

Unified shallow-water/Dupuit-Forchheimer model (*SW/DF*):

$$\partial_t \mathcal{V}(\eta) + \nabla\cdot\left(\mathcal{H}^s(\eta)\overline{u}^s\right) = \nabla\cdot\left(g\mathcal{V}^g(\eta)\overline{\kappa}^g(\eta)\nabla\eta\right)$$

$$\partial_t\left(\mathcal{H}^s(\eta)\overline{u}^s\right) + \nabla\cdot\left(\mathcal{H}^s(\eta)\overline{u}^s\otimes\overline{u}^s\right) = -g\mathcal{H}^s(\eta)\nabla\eta + \frac{\partial_t\mathcal{V}^s(\eta)+\nabla\cdot\left(\mathcal{V}^s(\eta)\overline{u}^s\right)}{2}\overline{u}^s$$

with $\mathcal{H}^s(\eta) := \max(0, \eta - B)$

Properties

- **Energy dissipation**: $\partial_t\left(\mathscr{P}(\eta)+\mathscr{K}(\eta,\overline{u}^s)\right)+\nabla\cdot\mathscr{G}(\eta,\overline{u}^s) = -\widetilde{\mathscr{K}}(\eta)$

  where $\mathscr{P}(\eta) := g\int_S^\eta \Theta z\,\mathrm{d}z\,, \qquad \mathscr{K}(\eta,\overline{u}^s) := \frac{\mathcal{H}^s(\eta)}{2}|\overline{u}^s|^2$

  $$\mathscr{G}(\eta,\overline{u}^s) := \left(g\eta\mathcal{H}^s(\eta)+\mathscr{K}(\eta,\overline{u}^s)\right)\overline{u}^s - g^2\eta\mathcal{V}^g(\eta)\overline{\kappa}^g\nabla\eta$$

  and $\widetilde{\mathscr{K}}(\eta) := g^2\mathcal{V}^g(\eta)\overline{\kappa}^g\left|\nabla\eta\right|^2.$

Unified Green-Naghdi/Hydrodynamic Dupuit-Forchheimer model (*GN/HDF*):

$$\partial_t\mathcal{V}(\eta) + \nabla\cdot\left(\mathcal{H}^s(\eta)\overline{u}^s\right) = \nabla\cdot\left(g\mathcal{V}^g(\eta)\left(\mathrm{I_d}+\mathscr{T}^{\overline{\kappa},\overline{\Theta}}_{S,\mathcal{H}^g(\eta)}\right)^{-1}\left(\overline{\kappa}^g(\eta)\nabla\eta\right)\right)$$

$$\left(\mathrm{I_d}+\mathscr{T}^{1,1}_{B,\mathcal{H}^s(\eta)}\right)^{-1}\partial_t\overline{u}^s + \overline{u}^s\cdot\nabla\overline{u}^s = -g\mathcal{H}^s(\eta)\nabla\eta - Q_{B,\eta}(\overline{u}^s)$$

Numerical scheme: ImEx scheme [Parisot, Vila'16] [Lteif, Kazolea, Parisot'25]

- **Implicit** continuity equation

$$\mathcal{V}\left(\eta_k^{n+1}\right)=\mathcal{V}\left(\eta_k^{n}\right)-\delta_t\nabla_k^{\delta}\cdot\left(\left\{\mathcal{H}^s\left(\eta_\star^{n+1}\right)\overline{u}_\star^n\right\}_\star-\left(\delta_t\left\{\gamma\left(\eta_\star^{n+1}\right)\right\}_\star+\left\{g\mathcal{V}^g\left(\eta_\star^{n+1}\right)\overline{\kappa}_\star^g\right\}_\star\right)\nabla_\star^{\delta}\left(\eta_\star^{n+1}\right)\right)$$

- **Explicit** vertical exchanges

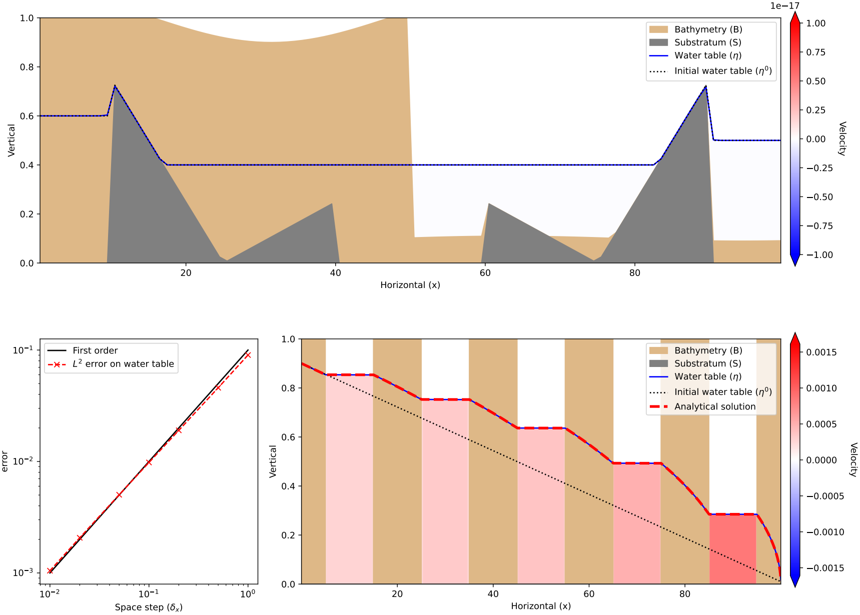
$$G_k^{n+1}=\frac{H_k^{s*}-\mathcal{H}^s\left(\eta_k^{n+1}\right)}{\delta_t}\qquad\text{with}\qquad H_k^{s*}=\mathcal{H}^s\left(\eta_k^n\right)-\delta_t\nabla_k^{\delta}\cdot\left(F_\star^{n+1}\right)$$

$$\text{and}\qquad F_f^{n+1}=\left\{\mathcal{H}^s\left(\eta_\star^{n+1}\right)\overline{u}_\star^n\right\}_f-\delta_t\left\{\gamma\left(\eta_\star^{n+1}\right)\right\}_f\nabla_f^{\delta}\left(\eta_\star^{n+1}\right)$$

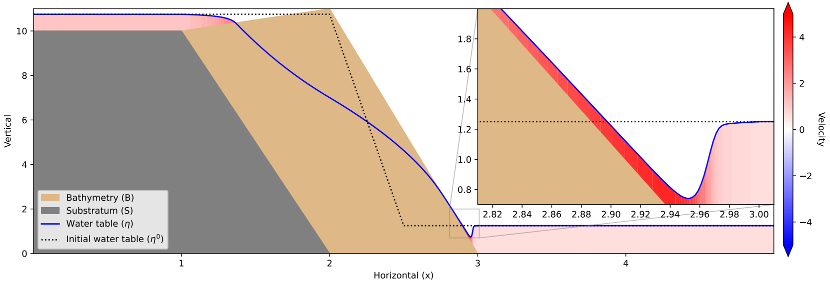
- **Explicit** momentum equation (si $\mathcal{V}^s\left(\eta_k^{n+1}\right)>0$)

$$\left(\mathcal{V}^s\left(\eta_k^{n+1}\right)+\frac{G_k^{n+1}}{2}\right)\overline{u}_k^{n+1}=\mathcal{V}^s\left(\eta_k^n\right)\overline{u}_k^n-\delta_t\left(\nabla_k^{\mathrm{up}}\cdot\left(\overline{u}_\star^n,F_\star^{n+1}\right)+g\mathcal{H}^s\left(\eta_k^{n+1}\right)\nabla_k^{\delta}\left(\eta_\star^{n+1}\right)\right)$$

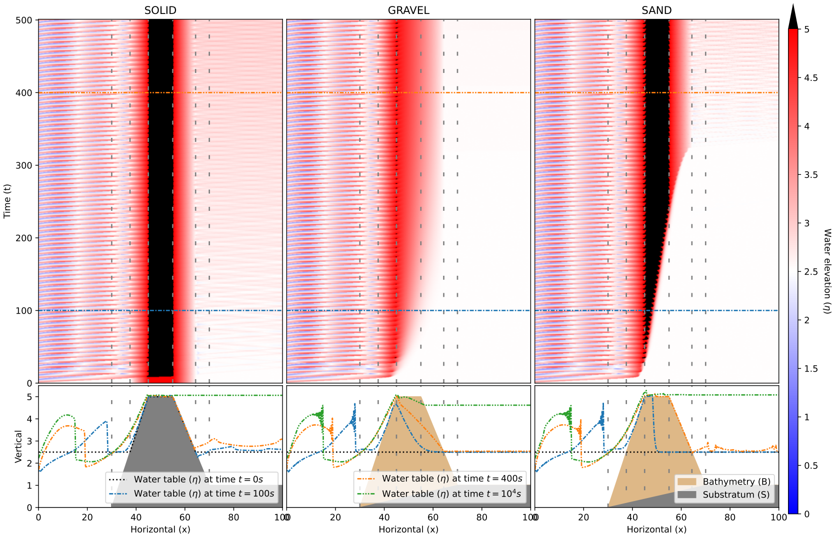
Entropy satisfying: for $\gamma(\eta)\geq g\mathcal{V}^s(\eta)$ and under a CFL condition,
the scheme satisfies a discrete counterpart of the energy dissipation.



Bathymetry (B)
Substratum (S)
Water table (η)
Initial water table (η⁰)
Vertical
Horizontal (x)
Velocity
1e−17
First order
L² error on water table
error
Space step (δx)
Analytical solution



Bathymetry (B)
Substratum (S)
Water table (η)
Initial water table (η^0)
Vertical
Horizontal (x)
Velocity



SOLID
GRAVEL
SAND
Time (t)
Vertical
Horizontal (x)
Water elevation (η)
Water table (η) at time t = 0s
Water table (η) at time t = 100s
Water table (η) at time t = 400s
Water table (η) at time t = 10^4 s
Bathymetry (B)
Substratum (S)

Conclusion:

- Derivation of an approximate **hydrodynamic** model [Parisot'25]
  - **improving** the linear decay rate
  - **preserving** the energy dissipation
- **Unified model** for shallow water and Dupuit-Forchheimer regime [Carreau,Parisot'25]
  - "coupling" between **hyperbolic** and **elliptic** models
  - **preserving** the energy dissipation

Perspectives:

- Analysis of the hydrodynamic model (maximum principle, monotony, TVD...)
- Analysis of the **numerical stability** (with/without CFL condition)
- **Higher order** hydrodynamic models
- **Vertical dependency** of the physical parameters κ and Θ
- **Infiltration model** in the vadose area *B. Chauveau, S. Erdocio* (IFPEN)
- Study of **interface conditions** *T. Bonnet, M. Coquerelle* (I2M)
- **Morphodynamic** coupling (physical parameters evolution, deformation)
- Same questions for the **Richards model** (scalar 3D advection-diffusion equation)

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Thank You