Modelling and dispersion relations of layer-averaged non-hydrostatic Euler equations

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Outline

- Introduction
- Laver-Averaged approximation of Navier-Stokes
- Dispersion relations

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- Dispersion relations

It is based on a splitting of the water column into an arbitrary number of layers which can be interpreted as a semi-discretisation along the vertical axis. Several definitions of the layers can be considered.

- First attempts (Miglio, Casulli) relied on (horizontal) layers of fixed thicknesses. In these cases, the free surface motion implied that the fluid domain does not coincide with the grid: some control volumes are then partially filled by the fluid. A similar problem occurs for control volumes crossing the bathymetry and which are also partially filled.
- Another possibility consists in considering the well-known σ-coordinates (Phillips).
- Another technique is the multilayer (or Layer-averaged) models, that was introduced under the assumption of hydrostatic pressures.
- An analogous multilayer discretisation was also considered in several papers in the literature (Lynett, Bai ...) for the case of hydrodynamic pressure.

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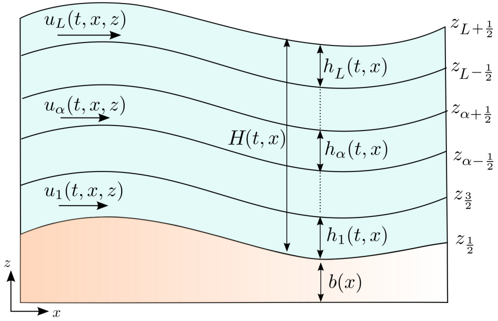
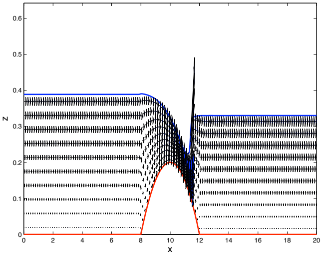


Figure: *Sketch of the multilayer division of the fluid domain.*

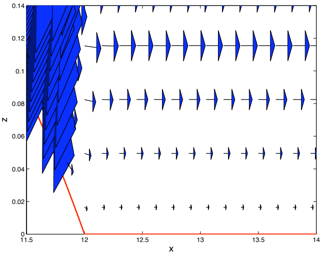
The fluid domain is divided along the vertical direction into $N \in \mathbb{N}^*$ pre-set layers of thickness $h_\alpha(t,x)$ with $N+1$ interfaces $\Gamma_{\alpha+\frac{1}{2}}(t)$ of equations $z = z_{\alpha+\frac{1}{2}}(t,x)$ for $\alpha = 0, 1, \ldots, N$ and $x \in I_F(t)$.



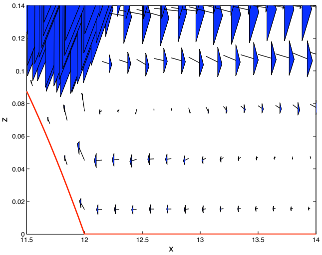
(a) Free surface and bottom.

Figure: Free surface evolution and velocity vectors

Stationary solution over a bump



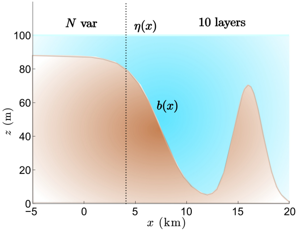
(a) $\nu = 0$

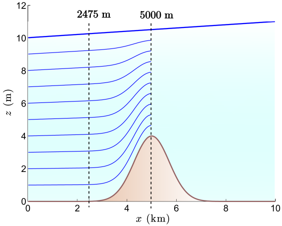


(b) $\nu = 10^{-3}$

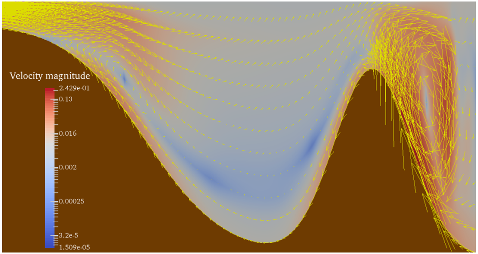
Figure: Test 1. Zoom at the right of the bump Left: Velocity vectors for $\nu = 0$. Right: Velocity vectors for $\nu = 10^{-3}$

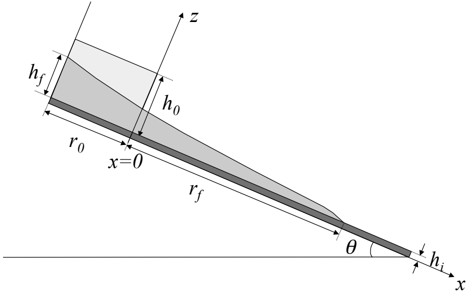
Bonaventura, Fernández-Nieto, Garres-Díaz, Narbona-Reina Multilayer shallow water models with locally variable number of layers and semi-implicit time discretization. JCP, 364, 209-234 (2018).





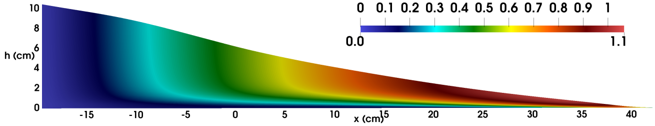
We **choose** a staggered mesh, although the multilayer approach can be employed for **any spatial discretization** in principle. We can also use **semi-implicit** methods.



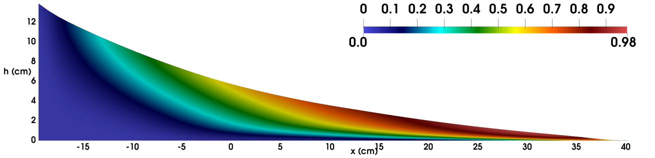

z
h_f
h_0
r_0
x=0
r_f
θ
h_i
x

Distribution of downslope velocity in a granular collapse ($\theta = 16^{\circ}$, $t = 0.3$ s):

$\mu(I)$ - multilayer



$\mu(I)$ - μ_w - multilayer



- The multilayer model with the side walls friction term allows us to better reproduce the normal structure of the flow.

Normal profiles of the downslope velocity (model with τ_{xz})

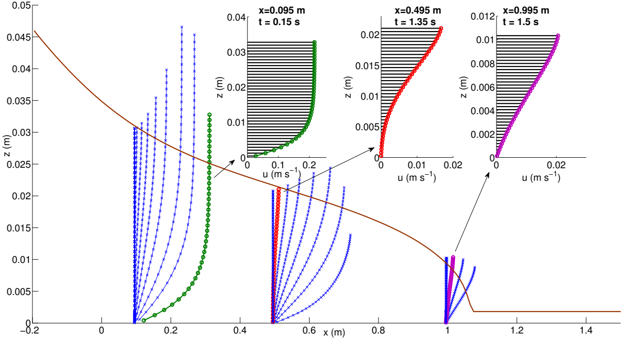
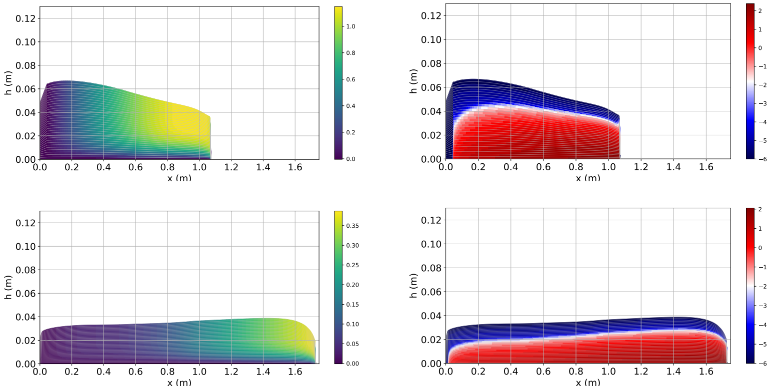


Figure: *Normal profiles of the downslope velocity obtained with the $\mu(I)$-MSM (40 layers) for $\theta = 22^\circ$ and $h_i = 1.82$ mm during granular collapse at different positions ($x = 0.095,\ 0.495,\ 0.995$ m).*

Viscoplastic dam break: vertical structure (model with τ_{xz})

$\theta = 15^\circ$ AT TIMES $t = 0.5, 1.5$ S (MODEL *without* $\tau_{xx},\ \tau_{zz}, N = 32$)

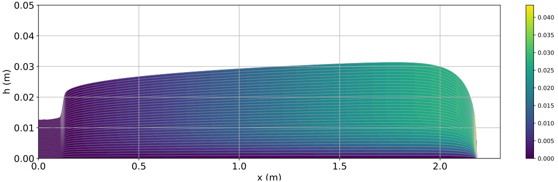
VELOCITY FIELD $\qquad\qquad$ $\log_{10}(|\partial_z u|)$



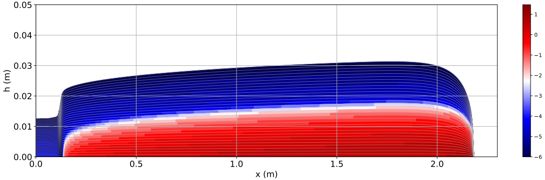
- Multilayer model reproduces the lower sheared layer ($||D|| > 0$) and top plug (unsheared) layer ($u(z)$ constant, $||D|| \approx 0$)

$\theta = 15^\circ$ AT TIME $t = 10$ S (MODEL *without* τ_{xx}, τ_{zz}, $N = 32$)

VELOCITY FIELD

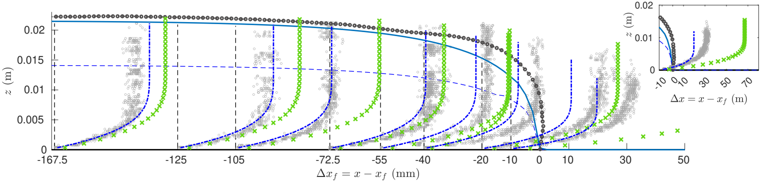


$\log_{10}(|\partial_z u|)$

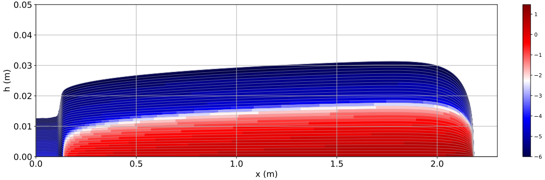


- Multilayer model reproduces the lower sheared layer ($||D|| > 0$) and top plug (unsheared) layer ($u(z)$ constant, $||D|| \approx 0$)

VERTICAL PROFILES OF VELOCITY AT DISTANCE Δx_f TO THE FRONT
MODEL WITHOUT τ_{xx}, τ_{zz} ($\theta = 25^\circ$, $t = 14.4$ s, $N = 32$)



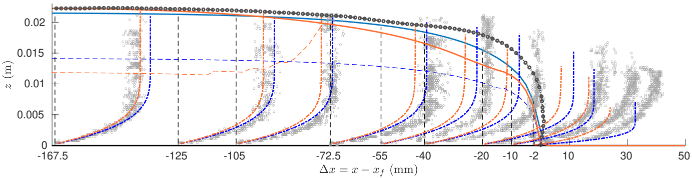
$\log_{10}(|\partial_z u|)$



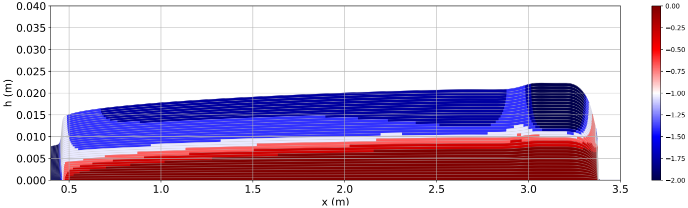
Gray symbols: experiments
Blue lines: multilayer results (dashed line: interface between sheared/plug layers)
Green crosses: analytical profiles with $\partial_x h$ computed from the experimental height

- Multilayer model reproduces the velocity profiles observed in the experiments
- In the experiments, a pseudoplug ($\|D\| \sim \mathcal{O}(\varepsilon)$) zone is observed instead of a plug ($\|D\| = 0$) zone
- In the experiments, we observe a change of curvature of the sheared/pseudoplug interface close to the front position

VERTICAL PROFILES OF VELOCITY AT DISTANCE Δx_f TO THE FRONT
MODEL WITH $\tau_{xx},\ \tau_{zz}$ ($\theta = 25^\circ$, $t = 14.4$ s, $N = 32$)



$\log_{10}(\|D_\varepsilon\|)$



- The model with normal stress components reproduces both the pseudoplug layer and the change of curvature of the interface

Outline

Let us consider a constant density $\rho \in \mathbb{R}$ for an incompressible fluid, which flows within the domain

$$\Omega(t) = \left\{(x,z) \in \mathbb{R}^2 : b(x) < z < b(x) + H(t,x)\right\},$$

being $b(x)$ and $H(t,x)$ a bottom topography and the total depth of the fluid.

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho\, \boldsymbol{U}) = 0, \\ \partial_t \left(\rho\, \boldsymbol{U}\right) + \nabla \cdot \left(\rho\, \boldsymbol{U} \otimes \boldsymbol{U}\right) \;-\; \nabla \cdot \boldsymbol{\sigma} = \rho\, \boldsymbol{g}, \end{cases}$$

where $\boldsymbol{\sigma} = -p\boldsymbol{I} + \boldsymbol{\tau}$

$$p = \rho\left(-g_z\left(b + H - z\right) + q\right), \qquad \boldsymbol{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xz} \\ \tau_{zx} & \tau_{zz} \end{pmatrix}.$$

with

$$\boldsymbol{g} = (g_x, g_z)' = \begin{cases} (0, -g)' & \text{in Cartesian coordinates,} \\ (-g\, \sin\theta, -g\, \cos\theta)' & \text{in local coordinates,} \end{cases}$$

where $g \in \mathbb{R}$ is the gravity acceleration. We also introduce the following notation:

$$z_b(x) = b(x) - \frac{g_x}{|g_z|}\, x.$$

$$(x,z,t) = (L\widetilde{x}, \mathcal{H}\widetilde{z}, (L/U)\widetilde{t}), \qquad H = \mathcal{H}\widetilde{H}, \qquad \rho = \rho_0\widetilde{\rho}, \qquad (u,w) = (U\widetilde{u}, \varepsilon U\widetilde{w}),$$

$$p = \rho_0 U^2 \widetilde{p}, \qquad (\tau_{xx}, \tau_{xz}, \tau_{zz}) = \rho_0 U^2 \left(\varepsilon\widetilde{\tau_{xx}}, \widetilde{\tau_{xz}}, \varepsilon\widetilde{\tau_{zz}}\right).$$

(We assume a flow regime where $|g_x| / \left(Fr^2|g_z|\right) \sim \mathcal{O}(1)$).

By defining the Froude number $Fr = U^2/\sqrt{|g_z|\mathcal{H}}$, the pressure is decomposed as

$$\widetilde{p} = \widetilde{\rho}\left(\frac{1}{Fr^2}\left(\widetilde{b} + \widetilde{H} - \widetilde{z}\right) + \varepsilon\widetilde{q}\right).$$

The non-dimensional form of the Navier-Stokes system reads

$$\begin{cases} \partial_x u + \partial_z w = 0, \\ \partial_t u + \partial_x(u^2) + \partial_z(u\,w) + \dfrac{1}{Fr^2}\partial_x(b+H) + \varepsilon\partial_x q = \dfrac{g_x}{\varepsilon Fr^2|g_z|} + \dfrac{1}{\rho}\left(\varepsilon\partial_x\tau_{xx} + \dfrac{1}{\varepsilon}\partial_z\tau_{xz}\right), \\ \varepsilon^2\left(\partial_t w + \partial_x(u\,w) + \partial_z(w^2)\right) + \varepsilon\partial_z q = \dfrac{\varepsilon}{\rho}\left(\partial_x\tau_{xz} + \partial_z\tau_{zz}\right), \end{cases}$$

(tildes are dropped for the sake of simplicity)

- Hydrostatic main order

$$\begin{cases} \partial_x u + \partial_z w = 0, \\ \partial_t u + \partial_x (u^2) + \partial_z (u\, w) + \dfrac{1}{Fr^2} \partial_x \left(b + H\right) = \dfrac{g_x}{\varepsilon Fr^2 |g_z|} + \dfrac{1}{\varepsilon \rho} \left(\partial_z \tau_{xz}\right). \end{cases}$$

- Hydrostatic first order
- Weakly non-hydrostatic
- Fully non-hydrostatic

- Hydrostatic main order

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"Asymptotic levels"

- Hydrostatic main order

- Hydrostatic first order

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$$\begin{cases} \partial_x u + \partial_z w = 0, \\ \partial_t u + \partial_x(u^2) + \partial_z(u\,w) + \dfrac{1}{Fr^2}\partial_x\left(b+H\right) + \varepsilon^2\partial_x(q_2) = \dfrac{g_x}{\varepsilon Fr^2|g_z|} + \dfrac{1}{\rho}\left(\dfrac{1}{\varepsilon}\partial_z\tau_{xz}\right), \\ \varepsilon^2\left(\partial_t w + \partial_x(u\,w) + \partial_z(w^2)\right) + \varepsilon^2\partial_z(q_2) = 0. \end{cases}$$

- Fully non-hydrostatic

Some references

Hydrostatic main order

E. D. Fernández-Nieto, J. Garres-Díaz, A. Mangeney, G. Narbona-Reina. (2016). A multilayer shallow model for dry granular flows with the $\mu(I)$-rheology: application to granular collapse on erodible beds. *Journal of Fluid Mechanics*, 798:643–681.

E. D. Fernández-Nieto, J. Garres-Díaz, A. Mangeney, G. Narbona-Reina. (2018). 2D granular flows with the $\mu(I)$ rheology and side walls friction: a well-balanced multilayer discretization. *Journal of Computational Physics*, 356:192–219.

E. D. Fernández-Nieto, J. Garres-Díaz, and P. Vigneaux (2023). Multilayer models for hydrostatic Herschel-Bulkley viscoplastic flows. *Computers & Mathematics with Applications*, 139:99–117.

Hydrostatic first order

M.-O. Bristeau, C. Guichard, B. Di Martino, J. Sainte-Marie. (2017). Layer-averaged Euler and Navier-Stokes equations *Communications in Mathematical Sciences*.

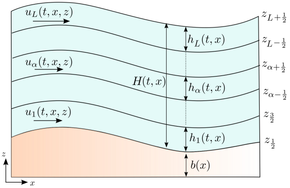
Weakly non-hydrostatic

J. Garres-Díaz, E. D. Fernández-Nieto, A. Mangeney, and T. Morales de Luna (2021). A weakly non-hydrostatic shallow model for dry granular flows. *Journal of Scientific Computing*, 86(2).

C. Escalante, E.D. Fernández-Nieto, J Garres-Díaz, A Mangeney (2023). Multilayer Shallow Model for Dry Granular Flows with a Weakly Non-hydrostatic Pressure. *Journal of Scientific Computing*, 96(3).

Fully non-hydrostatic

E.D. Fernández-Nieto, J Garres-Díaz (2023). Layer-averaged approximation of Navier–Stokes system with complex rheologies. *ESAIM: Mathematical Modelling and Numerical Analysis*, 57(5), 2735-2774.



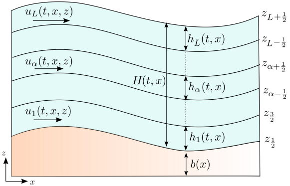
- We consider a subdivision in the vertical direction of the domain into $L \in \mathbb{N}$ shallow layers Ω_α, whose heights are h_α for $\alpha = 1, \ldots, L$.

$$\Omega_\alpha(t) = \left\{(x,z) \in \mathbb{R}^2 : z_{\alpha-1/2} < z < z_{\alpha+1/2}\right\},$$

where $z = z_{\alpha+1/2}$ defines the interface separating the layers Ω_α and $\Omega_{\alpha+1}$.

- The total height of the fluid is $H = \sum_{\beta=1}^{L} h_\beta$, and it holds that $h_\alpha = z_{\alpha+1/2} - z_{\alpha-1/2}$.
- The midpoint of each layer Ω_α is $z_\alpha = z_{\alpha-1/2} + h_\alpha/2$.
- The vertical mesh is defined through the coefficients $(\ell_\alpha)_{\{\alpha\in 1,\ldots,L\}}$ satisfying

$$h_\alpha = \ell_\alpha H, \quad \text{with} \quad \ell_\alpha \in [0,1] \quad \text{and} \quad \sum_{\alpha=1}^{L} \ell_\alpha = 1.$$



- For an arbitrary function $f(t,x,z)$. We denote by $f^{\pm}_{\alpha+1/2}$ its approximation at the interface

$$f^{-}_{\alpha+1/2} = \lim_{\substack{z\to z_{\alpha+1/2}\\ z<z_{\alpha+1/2}}} f_{|\Omega_\alpha}, \qquad f^{+}_{\alpha+1/2} = \lim_{\substack{z\to z_{\alpha+1/2}\\ z>z_{\alpha+1/2}}} f_{|\Omega_{\alpha+1}}.$$

We write $f_{\alpha+1/2}$ if both limits match and f is a continuous function.

- Averages: $\overline{f}_\alpha(t,x) = \dfrac{1}{h_\alpha}\displaystyle\int_{z_{\alpha-1/2}}^{z_{\alpha-1/2}} f(t,x,z)\mathrm{d}z$

$$\widehat{f}_\alpha = \frac{f^{+}_{\alpha-1/2} + f^{-}_{\alpha+1/2}}{2} \qquad \widetilde{f}_{\alpha+1/2} = \frac{f^{+}_{\alpha+1/2} + f^{-}_{\alpha+1/2}}{2}$$

- Variations:

$$(\delta f)_\alpha = f^{-}_{\alpha+1/2} - f^{+}_{\alpha-1/2} \qquad [f]_{\alpha+1/2} = f^{+}_{\alpha+1/2} - f^{-}_{\alpha+1/2}$$

Let us denote by

$$\boldsymbol{U}_\alpha := \boldsymbol{U}_{|\Omega_\alpha} := (u_\alpha, w_\alpha)',$$

the velocity in the layer Ω_α, where u_α and w_α are the horizontal and vertical components.

- We assume a linear profile in z for the horizontal velocity within each layer.

$$u_\alpha(z) = \overline{u}_\alpha + \lambda_\alpha \big(z - z_\alpha\big), \quad \text{for } z \in [z_{\alpha-1/2}, z_{\alpha+1/2}],$$

  - $\overline{u}_\alpha$ the averaged velocity
  - λ_α its slope.

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  - $\overline{u}_\alpha$ the averaged velocity
  - λ_α its slope.

Note: $u^-_{\alpha+1/2}$ and $u^+_{\alpha-1/2}$, the limit values of the horizontal velocity inside layer Ω_α at the interfaces $z_{\alpha+1/2}$ and $z_{\alpha-1/2}$, respectively, are given by

$$u^-_{\alpha+1/2} = \overline{u}_\alpha + \frac{h_\alpha \lambda_\alpha}{2}, \qquad u^+_{\alpha-1/2} = \overline{u}_\alpha - \frac{h_\alpha \lambda_\alpha}{2}.$$

Let us denote by

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the velocity in the layer Ω_α, where u_α and w_α are the horizontal and vertical components.

- For the vertical velocity, looking at the incompressibility condition, we consider a layerwise parabolic profile:

$$w_\alpha(z) = \overline{w}_\alpha + \varphi_\alpha(z - z_\alpha) + \frac{\psi_\alpha}{2}\left((z - z_\alpha)^2 - \frac{h_\alpha^2}{12}\right) \quad \text{for } z \in [z_{\alpha-1/2}, z_{\alpha+1/2}],$$

- The variables $\varphi_\alpha, \psi_\alpha$ can be related to the variables in the horizontal velocity by means of the incompressibility condition. Concretely, we obtain the constraints

$$\begin{cases} \varphi_\alpha = -\partial_x \overline{u}_\alpha + \lambda_\alpha \partial_x z_\alpha, \\ \psi_\alpha = -\partial_x \lambda_\alpha, \end{cases}$$

for $\alpha = 1, \ldots, L$.

Let us denote by

$$\boldsymbol{U}_\alpha := \boldsymbol{U}_{|\Omega_\alpha} := (u_\alpha, w_\alpha)',$$

the velocity in the layer Ω_α, where u_α and w_α are the horizontal and vertical components.

- For the vertical velocity, looking at the incompressibility condition, we consider a layerwise parabolic profile:

$$w_\alpha(z) = \overline{w}_\alpha + \varphi_\alpha(z - z_\alpha) + \frac{\psi_\alpha}{2}\left((z - z_\alpha)^2 - \frac{h_\alpha^2}{12}\right) \quad \text{for } z \in [z_{\alpha-1/2}, z_{\alpha+1/2}],$$

- The variables $\varphi_\alpha, \psi_\alpha$ can be related to the variables in the horizontal velocity by means of the incompressibility condition. Concretely, we obtain the constraints

$$\begin{cases} \varphi_\alpha = -\partial_x \overline{u}_\alpha + \lambda_\alpha \partial_x z_\alpha, \\ \psi_\alpha = -\partial_x \lambda_\alpha, \end{cases}$$

  for $\alpha = 1, \ldots, L$.

The limit values at the interfaces are then

$$w^-_{\alpha+1/2} = \overline{w}_\alpha + \frac{h_\alpha \varphi_\alpha}{2} + \frac{h_\alpha^2 \psi_\alpha}{12}, \qquad w^+_{\alpha-1/2} = \overline{w}_\alpha - \frac{h_\alpha \varphi_\alpha}{2} + \frac{h_\alpha^2 \psi_\alpha}{12}.$$

Concerning the non-hydrostatic pressure q, it is a layerwise cubic function accordingly to the vertical momentum equation.

- It is assumed to be continuous across the interfaces $(q_\alpha(z_{\alpha+1/2}) = q_{\alpha+1}(z_{\alpha+1/2}) = q_{\alpha+1/2})$.
- Let us consider the variable π_α satisfying

$$\partial_z q_\alpha(z_\alpha) = \frac{\pi_\alpha}{h_\alpha},$$

- Using the proposed notation, the vertical profile of non-hydrostatic pressure is

$$q_\alpha(z) = \frac{3\overline{q}_\alpha - \widehat{q}_\alpha}{2} + \pi_\alpha \frac{z - z_\alpha}{h_\alpha} + 6\left(\widehat{q}_\alpha - \overline{q}_\alpha\right) \frac{(z - z_\alpha)^2}{h_\alpha^2} + 4\big((\delta q)_\alpha - \pi_\alpha\big) \frac{(z - z_\alpha)^3}{h_\alpha^3}$$

for $z \in [z_{\alpha-1/2}, z_{\alpha+1/2}]$.

- Then, $q_\alpha(z)$ is defined in terms of

$$\overline{q}_\alpha, \qquad q_{\alpha\pm1/2} \qquad \text{and} \qquad \pi_\alpha.$$

We also assume a polynomial approximation, where the coefficient must be defined in terms of the considered rheology, in order to approximate the viscous terms appearing in the Navier-Stokes system

$$(\partial_x \tau_{xx}, \partial_z \tau_{xz}, \partial_x \tau_{xz}, \partial_z \tau_{zz}).$$

Let us consider the following definition of the layerwise stress tensor components,

$$\tau_{ij,\alpha}(z) = \overline{\tau}_{ij,\alpha} + \zeta_{ij,\alpha}(z - z_\alpha) + \xi_{ij,\alpha}\left(\frac{(z - z_\alpha)^2}{2} - \frac{h_\alpha^2}{24}\right) + \varkappa_{ij,\alpha}\left(\frac{(z - z_\alpha)^3}{3} - \frac{h_\alpha^2}{20}(z - z_\alpha)\right)$$

where $i, j \in \{x, z\}$ and we assume $\tau_{xz,\alpha} = \tau_{zx,\alpha}$.

Denoting the kinematic viscosity coefficient by ν, which could be variable, we consider the stress tensor given by

$$\boldsymbol{\tau} = \rho\nu D(\boldsymbol{U}) \qquad \text{where} \qquad D(\boldsymbol{U}) = \frac{1}{2}\left(\nabla\boldsymbol{U} + (\nabla\boldsymbol{U})'\right).$$

Therefore, its components are

$$\tau_{xx} = \rho\nu\partial_x u, \quad \tau_{xz} = \tau_{zx} = \rho\frac{\nu}{2}(\partial_x w + \partial_z u), \quad \tau_{zz} = \rho\nu\partial_z w.$$

Stress tensor proportional to the strain rate tensor: variable viscosity

- A goal of this work is to propose models for geophysical flows, which are represented by appropriate rheological laws.

  These laws can be defined through variable viscosity coefficients, which could depend, for instance, on the velocity and pressure.

  Therefore, the viscosity is also a function that must be approximated in the layer-averaged framework.

- We consider a linear approximation of the viscosity within each layer. Thus, it is defined by

  $$\nu_{ij,\alpha}(z) = \nu^0_{ij,\alpha} + \nu^1_{ij,\alpha}(z - z_\alpha), \qquad \text{for } z \in [z_{\alpha-1/2}, z_{\alpha+1/2}], \qquad i,j \in \{x,z\},$$

  and $\alpha = 1, \ldots, L$, such that $\nu_{ij,\alpha} \geq 0$.

- We will see that this approach is appropriate for several rheologies.

- We consider a linear approximation of the viscosity within each layer. Thus, it is defined by

$$\nu_{ij,\alpha}(z) = \nu^0_{ij,\alpha} + \nu^1_{ij,\alpha}(z - z_\alpha), \qquad \text{for } z \in [z_{\alpha-1/2}, z_{\alpha+1/2}], \qquad i,j \in \{x,z\},$$

and $\alpha = 1, \ldots, L$, such that $\nu_{ij,\alpha} \geq 0$.

- We will see that this approach is appropriate for several rheologies.

- Note that we are considering a different viscosity coefficient $\nu_{ij,\alpha}$ for each component of the deviatoric stress tensor, thus making broader the range of applicability of the proposed models. For instance, it is useful in the case of turbulent flows or, in general, when having different viscosity coefficients along the horizontal and vertical directions.

Stress tensor proportional to the strain rate tensor: $D(\boldsymbol{U})$

- A key point in the definition of the approximation of the stress tensor components is the approximation of $\partial_x u$, $\partial_x w$, $\partial_z u$ and $\partial_z w$ accounting for the possible discontinuities at the interfaces $z_{\alpha+1/2}$ of the velocity components u and w.

- Let us remind that, for a fixed time $t > 0$, for any vector function $\boldsymbol{F}(t,x,z) \in \Omega \subset \mathbb{R}^2$ being a regular solution within each layer Ω_α, for $\alpha = 1,\dots,L$, with possible discontinuities at the internal interfaces $\mathcal{L}_{\alpha+1/2}$, for $\alpha = 1,\dots,L-1$, we can define the divergence $[div_{(x,z)}\boldsymbol{F}(t,\cdot,\cdot)]$ in the sense of distributions

$$\langle [div_{(x,z)}\boldsymbol{F}(t,\cdot,\cdot)], \phi\rangle = \int_\Omega div_{(x,z)}\boldsymbol{F}(t,x,z)\phi(x,z)dxdz$$
$$+ \int_{I_\Omega} \sum_{\alpha=1}^{L-1} \left(\boldsymbol{F}^+_{\alpha+1/2} - \boldsymbol{F}^-_{\alpha+1/2}\right) \cdot \begin{pmatrix} -\partial_x z_{\alpha+1/2} \\ 1 \end{pmatrix} \phi\left(x, z_{\alpha+1/2}\right) dx$$

  $\forall \phi \in \mathcal{D}(\Omega)$, where:

  - $\boldsymbol{F}^\pm_{\alpha+1/2}(x)$ are the upper and lower limits of $\boldsymbol{F}(t,x,z)$ when z tends to $z_{\alpha+1/2}$, respectively.
  - $\mathcal{D}(\Omega)$ is the set of functions of class $C^\infty(\Omega)$ with compact support,
  - I_Ω the projection of Ω over $\mathbb{R}$, and
  - the divergence operator appearing in the double integral has to be understood in the pointwise sense.

$$\overline{[\partial_z u]}_\alpha = \overline{\partial_z (u_\alpha(z))}_\alpha + \frac{1}{h_\alpha}\frac{[u]_{\alpha+1/2}+[u]_{\alpha-1/2}}{2} = \lambda_\alpha + \frac{1}{h_\alpha}\frac{[u]_{\alpha+1/2}+[u]_{\alpha-1/2}}{2}$$

$$\overline{[\partial_z w]}_\alpha = \overline{\partial_z (w_\alpha(z))}_\alpha + \frac{1}{h_\alpha}\frac{[w]_{\alpha+1/2}+[w]_{\alpha-1/2}}{2} = \varphi_\alpha + \frac{1}{h_\alpha}\frac{[w]_{\alpha+1/2}+[w]_{\alpha-1/2}}{2}$$

- These definitions can be seen as a partition of $\langle div_{(x,z)}\boldsymbol{F}, \mathbf{1}_\Omega\rangle$ for $\boldsymbol{F} = (0,u)'$, $\boldsymbol{F} = (0,w)'$

,

$$\langle [\partial_z u(t,\cdot,\cdot)], \mathbf{1}_\Omega\rangle = \sum_{\alpha=1}^{L}\int_{I_\Omega} h_\alpha \overline{[\partial_z u]}_\alpha dx \qquad \langle [\partial_z w(t,\cdot,\cdot)], \mathbf{1}_\Omega\rangle = \sum_{\alpha=1}^{L}\int_{I_\Omega} h_\alpha \overline{[\partial_z w]}_\alpha dx$$

$$
\begin{aligned}
\overline{[\partial_x u]}_\alpha &= \overline{\partial_x(u_\alpha(z))}_\alpha - \frac{1}{h_\alpha}\frac{[u]_{\alpha+1/2}\partial_x z_{\alpha+1/2} + [u]_{\alpha-1/2}\partial_x z_{\alpha-1/2}}{2} = \\
&= \partial_x \overline{u}_\alpha - \lambda_\alpha \partial_x z_\alpha - \frac{1}{h_\alpha}\frac{[u]_{\alpha+1/2}\partial_x z_{\alpha+1/2} + [u]_{\alpha-1/2}\partial_x z_{\alpha-1/2}}{2}
\end{aligned}
$$

$$
\begin{aligned}
\overline{[\partial_x w]}_\alpha &= \overline{\partial_x(w_\alpha(z))}_\alpha - \frac{1}{h_\alpha}\frac{[w]_{\alpha+1/2}\partial_x z_{\alpha+1/2} + [w]_{\alpha-1/2}\partial_x z_{\alpha-1/2}}{2} \\
&= \partial_x \overline{w}_\alpha - \varphi_\alpha \partial_x z_\alpha - \frac{h_\alpha \psi_\alpha}{12}\partial_x h_\alpha - \frac{1}{h_\alpha}\frac{[w]_{\alpha+1/2}\partial_x z_{\alpha+1/2} + [w]_{\alpha-1/2}\partial_x z_{\alpha-1/2}}{2}
\end{aligned}
$$

- These definitions can be seen as a partition of $\langle div_{(x,z)}\boldsymbol{F}, \mathbf{1}_\Omega\rangle$ for $\boldsymbol{F} = (u,0)$, and $\boldsymbol{F} = (w,0)$,

´

$$
\langle [\partial_x u(t,\cdot,\cdot)], \mathbf{1}_\Omega\rangle = \sum_{\alpha=1}^{L}\int_{I_\Omega} h_\alpha \overline{[\partial_x u]}_\alpha dx \qquad \langle [\partial_x w(t,\cdot,\cdot)], \mathbf{1}_\Omega\rangle = \sum_{\alpha=1}^{L}\int_{I_\Omega} h_\alpha \overline{[\partial_x w]}_\alpha dx
$$

Taking into account the linear profile of the viscosity (26), τ_{xx} is approximated at each layer by $\tau_{xx,\alpha}$, where we set

$$\tau_{xx,\alpha}(z) = \left(\nu^0_{xx,\alpha} + \nu^1_{xx,\alpha}\,(z - z_\alpha)\right)\left(\overline{[\partial_x u]}_\alpha + \partial_x\lambda_\alpha\,(z - z_\alpha)\right)$$

Note that it can be rewritten under the form

$$\tau_{xx,\alpha}(z) = \overline{\tau}_{xx,\alpha} + \zeta_{xx,\alpha}(z - z_\alpha) + \xi_{xx,\alpha}\left(\frac{(z-z_\alpha)^2}{2} - \frac{h_\alpha^2}{24}\right) + \varkappa_{xx,\alpha}\left(\frac{(z-z_\alpha)^3}{3} - \frac{h_\alpha^2}{20}(z - z_\alpha)\right)$$

with the following definition of the components:

$$\begin{aligned}
\overline{\tau}_{xx,\alpha} &= \frac{1}{h_\alpha}\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \tau_{xx,\alpha}(z)dz = \nu^0_{xx,\alpha}\overline{[\partial_x u]}_\alpha + \nu^1_{xx,\alpha}\frac{h_\alpha^2}{12}\partial_x\lambda_\alpha,\\
\zeta_{xx,\alpha} &= \nu^1_{xx,\alpha}\overline{[\partial_x u]}_\alpha + \nu^0_{xx,\alpha}\partial_x\lambda_\alpha,\\
\xi_{xx,\alpha} &= 2\nu^1_{xx,\alpha}\partial_x\lambda_\alpha,\\
\varkappa_{xx,\alpha} &= 0.
\end{aligned}$$

The approximation of τ_{zz} at each layer given by

$$\tau_{zz,\alpha}(z) = \left(\nu^0_{zz,\alpha} + \nu^1_{zz,\alpha}(z - z_\alpha)\right)\left(\overline{[\partial_z w]}_\alpha + \psi_\alpha(z - z_\alpha)\right).$$

For the approximation of τ_{zz} at each layer, we consider

$$\tau_{xz,\alpha}(z) = \frac{\nu^0_{xz,\alpha} + \nu^1_{xz,\alpha}(z - z_\alpha)}{2}\Bigg(\overline{[\partial_z u]}_\alpha + \overline{[\partial_x w]}_\alpha$$
$$+ (\partial_x \varphi_\alpha - \psi_\alpha \partial_x z_\alpha)(z - z_\alpha) + \partial_x \psi_\alpha \left(\frac{(z - z_\alpha)^2}{2} - \frac{h_\alpha^2}{24}\right)\Bigg).$$

$$\partial_t H + \partial_x \left(H\overline{\overline{u}}\right) = 0$$

with

$$\overline{\overline{u}} = \sum_{\alpha=1}^{L} \ell_\alpha \overline{u}_\alpha$$

$$\partial_t H + \partial_x \left(H\overline{\overline{u}}\right) = 0$$

$$\partial_t \left(h_\alpha \overline{u}_\alpha\right) + \partial_x \left(h_\alpha \overline{u}_\alpha^2 + \frac{h_\alpha^3 \lambda_\alpha^2}{12}\right) + |g_z|\, h_\alpha \partial_x \left(z_b + H\right) + \partial_x \left(h_\alpha \overline{q}_\alpha\right)$$
$$= q_{\alpha+1/2}\partial_x z_{\alpha+1/2} - q_{\alpha-1/2}\partial_x z_{\alpha-1/2} + \partial_x \left(h_\alpha \frac{\overline{\tau}_{xx,\alpha}}{\rho}\right) + K_{\alpha-1/2} - K_{\alpha+1/2}$$
$$+\widetilde{u}_{\alpha-1/2}\Gamma_{\alpha-1/2} - \widetilde{u}_{\alpha+1/2}\Gamma_{\alpha+1/2},$$

$$\partial_t H + \partial_x \left(H\overline{\overline{u}}\right) = 0$$

$$\partial_t \left(h_\alpha \overline{u}_\alpha\right) + \partial_x\left(h_\alpha \overline{u}_\alpha^2 + \frac{h_\alpha^3 \lambda_\alpha^2}{12}\right) + |g_z|\, h_\alpha \partial_x \left(z_b + H\right) + \partial_x \left(h_\alpha \overline{q}_\alpha\right)$$
$$= q_{\alpha+1/2}\partial_x z_{\alpha+1/2} - q_{\alpha-1/2}\partial_x z_{\alpha-1/2} + \partial_x\left(h_\alpha \frac{\overline{\tau}_{xx,\alpha}}{\rho}\right) + K_{\alpha-1/2} - K_{\alpha+1/2}$$
$$+\widetilde{u}_{\alpha-1/2}\Gamma_{\alpha-1/2} - \widetilde{u}_{\alpha+1/2}\Gamma_{\alpha+1/2},$$

At the interfaces, the approximation of the components of the normal projection of the stress tensor is

$$K_{\alpha+1/2} = \frac{1}{\rho}\left(\frac{\overline{\tau}_{xx,\alpha} + \overline{\tau}_{xx,\alpha+1}}{2}\,\partial_x z_{\alpha+1/2} - \frac{\overline{\tau}_{xz,\alpha} + \overline{\tau}_{xz,\alpha+1}}{2}\right),$$
$$K_{w,\alpha+1/2} = \frac{1}{\rho}\left(\frac{\overline{\tau}_{xz,\alpha} + \overline{\tau}_{xz,\alpha+1}}{2}\,\partial_x z_{\alpha+1/2} - \frac{\overline{\tau}_{zz,\alpha} + \overline{\tau}_{zz,\alpha+1}}{2}\right),$$

Where $K_{L+1/2} = 0$ and $K_{1/2}$ is defined by a friction law, for example,

$$K_{1/2} = -\frac{1}{\rho}\sqrt{1 + (\partial_x b)^2}\,\beta\left(\boldsymbol{U}_{1/2}^{+}\right) u_{1/2}^{+}, \qquad \beta\left(\left|\boldsymbol{U}_{1/2}^{+}\right|\right) = \beta_0 + \frac{\beta_1}{\left|\boldsymbol{U}_{1/2}^{+}\right|},$$

$$\partial_t H + \partial_x \left(H \overline{\overline{u}} \right) = 0$$

$$\partial_t \left(h_\alpha \overline{u}_\alpha \right) + \partial_x \left(h_\alpha \overline{u}_\alpha^2 + \frac{h_\alpha^3 \lambda_\alpha^2}{12} \right) + |g_z| \, h_\alpha \partial_x \left(z_b + H \right) + \partial_x \left(h_\alpha \overline{q}_\alpha \right)$$
$$= q_{\alpha+1/2} \partial_x z_{\alpha+1/2} - q_{\alpha-1/2} \partial_x z_{\alpha-1/2} + \partial_x \left(h_\alpha \frac{\overline{\tau}_{xx,\alpha}}{\rho} \right) + K_{\alpha-1/2} - K_{\alpha+1/2}$$
$$+ \widetilde{u}_{\alpha-1/2} \Gamma_{\alpha-1/2} - \widetilde{u}_{\alpha+1/2} \Gamma_{\alpha+1/2},$$

$\Gamma_{\alpha+1/2}$ denotes a explicit expression for the mass transference term at the interfaces $\mathcal{L}_{\alpha+1/2}$ in terms of the velocities and the fluid depth

$$\Gamma_{\alpha+1/2} = \sum_{\beta=\alpha+1}^{L} \ell_\beta \partial_x \left(H \left(\overline{u}_\beta - \overline{\overline{u}} \right) \right), \qquad \text{for } \alpha = 1, \ldots, L-1.$$

$$\partial_t H + \partial_x \left(H\overline{\overline{u}}\right) = 0$$

$$\partial_t \left(h_\alpha \overline{u}_\alpha\right) + \partial_x \left(h_\alpha \overline{u}_\alpha^2 + \frac{h_\alpha^3 \lambda_\alpha^2}{12}\right) + |g_z|\, h_\alpha \partial_x \left(z_b + H\right) + \partial_x \left(h_\alpha \overline{q}_\alpha\right)$$
$$= q_{\alpha+1/2}\partial_x z_{\alpha+1/2} - q_{\alpha-1/2}\partial_x z_{\alpha-1/2} + \partial_x \left(h_\alpha \frac{\overline{\tau}_{xx,\alpha}}{\rho}\right) + K_{\alpha-1/2} - K_{\alpha+1/2}$$
$$+\widetilde{u}_{\alpha-1/2}\Gamma_{\alpha-1/2} - \widetilde{u}_{\alpha+1/2}\Gamma_{\alpha+1/2},$$

$$\partial_t \left(\frac{h_\alpha^2 \lambda_\alpha}{12}\right) + \partial_x \left(\frac{h_\alpha^2 \lambda_\alpha \overline{u}_\alpha}{12} + \frac{h_\alpha (q_{\alpha+1/2} - q_{\alpha-1/2})}{20} + \frac{h_\alpha \pi_\alpha}{30}\right) + \frac{h_\alpha^2 \lambda_\alpha}{12}\partial_x \overline{u}_\alpha$$
$$+\overline{q}_\alpha \partial_x z_\alpha + \left(\frac{(q_{\alpha+1/2} - q_{\alpha-1/2})}{20} + \frac{\pi_\alpha}{30}\right)\partial_x h_\alpha$$
$$= \frac{1}{2}\left(q_{\alpha+1/2}\partial_x z_{\alpha+1/2} + q_{\alpha-1/2}\partial_x z_{\alpha-1/2}\right)$$
$$+\partial_x \left(\frac{h_\alpha^2 \zeta_{xx,\alpha}}{12\rho}\right) + \frac{h_\alpha \zeta_{xx,\alpha}}{12\rho}\partial_x h_\alpha + \frac{1}{\rho}\left(\overline{\tau}_{xx,\alpha}\partial_x z_\alpha - \overline{\tau}_{xz,\alpha}\right) - \frac{1}{2}\left(K_{\alpha+1/2} + K_{\alpha-1/2}\right)$$
$$-\Gamma_{\alpha-1/2}\left(\frac{h_\alpha \lambda_\alpha}{12} - \frac{\overline{u}_\alpha - \widetilde{u}_{\alpha-1/2}}{2}\right) + \Gamma_{\alpha+1/2}\left(\frac{h_\alpha \lambda_\alpha}{12} + \frac{\overline{u}_\alpha - \widetilde{u}_{\alpha+1/2}}{2}\right),$$

$$\partial_t \left(h_\alpha \overline{w}_\alpha\right) + \partial_x \left(h_\alpha \overline{u}_\alpha \overline{w}_\alpha + \frac{h_\alpha^3 \varphi_\alpha \lambda_\alpha}{12}\right) = q_{\alpha-1/2} - q_{\alpha+1/2}$$
$$+\partial_x \left(h_\alpha \frac{\overline{\tau}_{xz,\alpha}}{\rho}\right) + K_{w,\alpha-1/2} - K_{w,\alpha+1/2} + \widetilde{w}_{\alpha-1/2}\Gamma_{\alpha-1/2} - \widetilde{w}_{\alpha+1/2}\Gamma_{\alpha+1/2},$$

$$K_{w,\alpha+1/2} = \frac{1}{\rho}\left(\frac{\overline{\tau}_{xz,\alpha} + \overline{\tau}_{xz,\alpha+1}}{2}\, \partial_x z_{\alpha+1/2} - \frac{\overline{\tau}_{zz,\alpha} + \overline{\tau}_{zz,\alpha+1}}{2} \right)$$

Layer-averaged LIN-NH$_2$-STRESS model

$$
\partial_t\left(h_\alpha \overline{w}_\alpha\right)+\partial_x\left(h_\alpha \bar{u}_\alpha \overline{w}_\alpha+\frac{h_\alpha^3 \varphi_\alpha \lambda_\alpha}{12}\right)=q_{\alpha-1/2}-q_{\alpha+1/2}
$$
$$
+\partial_x\left(h_\alpha \frac{\overline{\tau}_{xz,\alpha}}{\rho}\right)+K_{w,\alpha-1/2}-K_{w,\alpha+1/2}+\widetilde{w}_{\alpha-1/2}\Gamma_{\alpha-1/2}-\widetilde{w}_{\alpha+1/2}\Gamma_{\alpha+1/2},
$$

$$
\partial_t\left(\frac{h_\alpha^2 \varphi_\alpha}{12}\right)+\partial_x\left(\frac{h_\alpha^2 \varphi_\alpha \bar{u}_\alpha}{12}+\frac{h_\alpha^4 \lambda_\alpha \psi_\alpha}{360}\right)+\frac{h_\alpha^2 \lambda_\alpha}{12}\partial_x \overline{w}_\alpha-\frac{h_\alpha^4 \psi_\alpha^2}{720}
$$
$$
+\frac{h_\alpha^3 \lambda_\alpha \psi_\alpha}{360}\partial_x h_\alpha+\frac{q_{\alpha+1/2}+q_{\alpha-1/2}}{2}-\overline{q}_\alpha=\partial_x\left(\frac{h_\alpha^2 \zeta_{xz,\alpha}}{12\rho}\right)+\frac{h_\alpha \zeta_{xz,\alpha}}{12\rho}\partial_x h_\alpha
$$
$$
+\frac{1}{\rho}\left(\overline{\tau}_{xz,\alpha}\partial_x z_\alpha-\overline{\tau}_{zz,\alpha}\right)-\frac{1}{2}\left(K_{w,\alpha-1/2}+K_{w,\alpha+1/2}\right)
$$
$$
-\Gamma_{\alpha-1/2}\left(\frac{h_\alpha \varphi_\alpha}{12}-\frac{\overline{w}_\alpha-\widetilde{w}_{\alpha-1/2}}{2}\right)+\Gamma_{\alpha+1/2}\left(\frac{h_\alpha \varphi_\alpha}{12}+\frac{\overline{w}_\alpha-\widetilde{w}_{\alpha+1/2}}{2}\right),
$$

$$\partial_t\left(h_\alpha \overline{w}_\alpha\right)+\partial_x\left(h_\alpha \overline{u}_\alpha \overline{w}_\alpha+\frac{h_\alpha^3\varphi_\alpha\lambda_\alpha}{12}\right)=q_{\alpha-1/2}-q_{\alpha+1/2}$$
$$+\partial_x\left(h_\alpha\frac{\overline{\tau}_{xz,\alpha}}{\rho}\right)+K_{w,\alpha-1/2}-K_{w,\alpha+1/2}+\widetilde{w}_{\alpha-1/2}\Gamma_{\alpha-1/2}-\widetilde{w}_{\alpha+1/2}\Gamma_{\alpha+1/2},$$

$$\partial_t\left(\frac{h_\alpha^2\varphi_\alpha}{12}\right)+\dots$$

$$\partial_t\left(\frac{h_\alpha^3\psi_\alpha}{720}\right)+\partial_x\left(\frac{h_\alpha^3\psi_\alpha\overline{u}_\alpha}{720}+\frac{h_\alpha^3\varphi_\alpha\lambda_\alpha}{360}\right)-\frac{h_\alpha^2\lambda_\alpha\varphi_\alpha}{120}\partial_x h_\alpha+\frac{h_\alpha^3\psi_\alpha\varphi_\alpha}{240}$$
$$=-\frac{(q_{\alpha+1/2}-q_{\alpha-1/2})-\pi_\alpha}{30}+\partial_x\left(\frac{h_\alpha^3\xi_{xz,\alpha}}{720\rho}\right)+\frac{h_\alpha^2\xi_{xz,\alpha}}{360\rho}\partial_x h_\alpha$$
$$+\frac{h_\alpha}{12\rho}\left(\zeta_{xz,\alpha}\partial_x z_\alpha-\zeta_{zz,\alpha}\right)+\frac{\overline{\tau}_{xz,\alpha}}{12\rho}\partial_x h_\alpha-\frac{1}{12}\left(K_{w,\alpha+1/2}-K_{w,\alpha-1/2}\right)$$
$$-\Gamma_{\alpha-1/2}\left(\frac{h_\alpha^2\psi_\alpha}{360}-\frac{h_\alpha\varphi_\alpha}{24}+\frac{\overline{w}_\alpha-\widetilde{w}_{\alpha-1/2}}{12}\right)$$
$$+\Gamma_{\alpha+1/2}\left(\frac{h_\alpha^2\psi_\alpha}{360}+\frac{h_\alpha\varphi_\alpha}{24}+\frac{\overline{w}_\alpha-\widetilde{w}_{\alpha+1/2}}{12}\right)$$

Combined with the following constraints,

$$
\begin{cases}
\varphi_\alpha = -\partial_x \overline{u}_\alpha + \lambda_\alpha \partial_x z_\alpha, & \alpha = 1, \ldots, L. \\[2ex]
\psi_\alpha = -\partial_x \lambda_\alpha, & \alpha = 1, \ldots, L, \\[2ex]
\overline{w}_{\alpha+1} - \dfrac{h_{\alpha+1}\varphi_{\alpha+1}}{2} + \dfrac{h_{\alpha+1}^2 \psi_{\alpha+1}}{12} - \overline{w}_\alpha - \dfrac{h_\alpha \varphi_\alpha}{2} - \dfrac{h_\alpha^2 \psi_\alpha}{12} & \\
\qquad = \left(\overline{u}_{\alpha+1} - \dfrac{h_{\alpha+1}\lambda_{\alpha+1}}{2} - \overline{u}_\alpha - \dfrac{h_\alpha \lambda_\alpha}{2} \right) \partial_x z_{\alpha+1/2}, & \alpha = 1, \ldots, L-1. \\[2ex]
\partial_x \overline{u}_1 - \lambda_1 \partial_x z_1 - \dfrac{h_1}{6} \partial_x \lambda_1 + \dfrac{\overline{w}_1 - w_{1/2}^+}{h_1/2} = 0. &
\end{cases}
$$

We consider the following notation for the velocity unknowns:

$$\Lambda_\alpha = \frac{h_\alpha \lambda_\alpha}{2\sqrt{3}}, \quad \Phi_\alpha = \frac{h_\alpha \varphi_\alpha}{2\sqrt{3}}, \quad \Psi_\alpha = \frac{h_\alpha^2 \psi_\alpha}{12\sqrt{5}},$$

and we set the following notation for the stress tensor components,

$$\mathcal{Z}_{xx,\alpha} = \frac{h_\alpha \zeta_{xx,\alpha}}{2\sqrt{3}}, \quad \mathcal{Z}_{xz,\alpha} = \frac{h_\alpha \zeta_{xz,\alpha}}{2\sqrt{3}}, \quad \mathcal{R}_{xz,\alpha} = \frac{h_\alpha \xi_{xz,\alpha}}{12\sqrt{5}}.$$

$$\begin{cases} \partial_t H + \partial_x \left(H\overline{\overline{u}} \right) = 0, \\ \partial_t (h_\alpha \mathbf{X}_\alpha) + \partial_x (h_\alpha \mathbf{X}_\alpha \overline{u}_\alpha) + \mathbf{F}_\alpha + \nabla_{NH} \mathbf{Q}_\alpha = \mathbf{S}_\alpha \partial_x \left(z_b + H \right) + \partial_x \mathbf{D}_{\tau,\alpha} \\ \qquad\qquad + \Gamma_{\alpha+1/2} \mathbf{G}^+_\alpha - \Gamma_{\alpha-1/2} \mathbf{G}^-_\alpha + \mathbf{G}^+_{\tau,\alpha} - \mathbf{G}^-_{\tau,\alpha}, \\ \nabla_{NH} \cdot \mathbf{X}_\alpha = 0, \end{cases}$$

where

$$\mathbf{X}_\alpha = \begin{pmatrix} \overline{u}_\alpha \\ \Lambda_\alpha \\ \overline{w}_\alpha \\ \phi_\alpha \\ \psi_\alpha \end{pmatrix}, \qquad \mathbf{Q}_\alpha = \begin{pmatrix} \overline{q}_\alpha \\ q_{\alpha-1/2} \\ \pi_\alpha \end{pmatrix} \qquad \mathbf{D}_{\tau,\alpha} = \frac{1}{\rho} \begin{pmatrix} h_\alpha \overline{\tau}_{xx,\alpha} \\ h_\alpha \mathcal{Z}_{xx,\alpha} \\ h_\alpha \overline{\tau}_{xz,\alpha} \\ h_\alpha \mathcal{Z}_{xz,\alpha} \\ h_\alpha \mathcal{R}_{xz,\alpha} \end{pmatrix},$$

and

$$\mathbf{G}^\pm_{\tau,\alpha} = \frac{1}{\rho} \begin{pmatrix} -\rho K_{\alpha\pm1/2} \\ \mathcal{Z}_{xx,\alpha} \partial_x z_{\alpha\pm1/2} \pm \sqrt{3} \left((\overline{\tau}_{xx,\alpha} \partial_x z_\alpha - \overline{\tau}_{xz,\alpha}) - \rho K_{\alpha\pm1/2} \right) \\ -\rho K_{w,\alpha\pm1/2} \\ \mathcal{Z}_{xz,\alpha} \partial_x z_{\alpha\pm1/2} \pm \sqrt{3} \left((\overline{\tau}_{xz,\alpha} \partial_x z_\alpha - \overline{\tau}_{zz,\alpha}) - \rho K_{w,\alpha\pm1/2} \right) \\ 2\mathcal{R}_{xz,\alpha} \pm \sqrt{15} \left(\mathcal{Z}_{xz,\alpha} \partial_x z_\alpha - \mathcal{Z}_{zz,\alpha} \right) + \sqrt{5} \left(\overline{\tau}_{xz,\alpha} \partial_x z_{\alpha\pm1/2} - \rho K_{w,\alpha\pm1/2} \right) \end{pmatrix},$$

Theorem (Energy balance)

$$
\begin{aligned}
&\partial_t \left(\sum_{\alpha=1}^{N} E_\alpha \right) + \partial_x \left[\sum_{\alpha=1}^{N} \left(\overline{u}_\alpha \left(E_\alpha + |g_z|\, h_\alpha \frac{h}{2} + \frac{h_\alpha^3 \lambda_\alpha^2}{12} + h_\alpha \overline{q}_\alpha \right) + \frac{h_\alpha^3 \lambda_\alpha \varphi_\alpha \overline{w}_\alpha}{12} \right.\right. \\
&\quad + \frac{h_\alpha^5 \lambda_\alpha \varphi_\alpha \psi_\alpha}{360} + \lambda_\alpha \left(\frac{h_\alpha^2 \pi_\alpha}{30} + \frac{h_\alpha^2 (q_{\alpha+1/2} - q_{\alpha-1/2})}{20} \right) \\
&\quad \left.\left. - \frac{1}{\rho} \left(h_\alpha \overline{u}_\alpha \overline{\tau}_{xx,\alpha} + \frac{h_\alpha^3 \lambda_\alpha \zeta_{xx,\alpha}}{12} + h_\alpha \overline{w}_\alpha \overline{\tau}_{xz,\alpha} + \frac{h_\alpha^3 \varphi_\alpha \zeta_{xz,\alpha}}{12} + \frac{h_\alpha^5 \psi_\alpha \xi_{xz,\alpha}}{720} \right) \right) \right] \\
&\quad \le -\frac{1}{\rho} \sum_{\alpha=1}^{N} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \left[\tau_{xx,\alpha}(z) \left(\overline{[\partial_x u]} + \partial_x \lambda_\alpha (z - z_\alpha) \right) \right. \\
&\quad + \tau_{xz,\alpha}(z) \left(\overline{[\partial_z u]}_\alpha + \overline{[\partial_x w]} + (\partial_x \varphi_\alpha - \psi_\alpha \partial_x z_\alpha)(z - z_\alpha) \right. \\
&\qquad \left.\left. + \partial_x \psi_\alpha \left(\frac{(z - z_\alpha)^2}{2} - \frac{h_\alpha^2}{24} \right) \right) + \tau_{zz,\alpha}(z) \left(\overline{[\partial_z w]}_\alpha + \psi_\alpha (z - z_\alpha) \right) \right] dz \\
&\quad - \frac{1}{\rho} \left(\beta_0 + \frac{\beta_1}{|\boldsymbol{U}|} \right) \left(1 + (\partial_x b)^2 \right)^{3/2} \left(u_{1/2}^{+} \right)^2 .
\end{aligned}
$$

where

$$
E_\alpha := h_\alpha \left(\frac{\overline{u}_\alpha^2 + \overline{w}_\alpha^2}{2} + \frac{(h_\alpha \lambda_\alpha)^2 + (h_\alpha \varphi_\alpha)^2}{24} + \frac{(h_\alpha^2 \psi_\alpha)^2}{1440} + |g_z| \left(z_b + \frac{h}{2} \right) \right)
$$

Corollary

Let us consider a stress tensor proportional to the strain rate tensor. Then, LIN-NH$_2$-STRESS model satisfies the dissipative energy balance, where the right-hand side is non-positive, being

$$
\begin{aligned}
&-\frac{1}{\rho}\sum_{\alpha=1}^{N}\int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}}\Bigg[\left(\nu^0_{xx,\alpha}+\nu^1_{xx,\alpha}(z-z_\alpha)\right)\left(\overline{[\partial_x u]}+\partial_x\lambda_\alpha(z-z_\alpha)\right)^2\\
&+\frac{(\nu^0_{xz,\alpha}+\nu^1_{xz,\alpha}(z-z_\alpha))}{2}\Bigg(\overline{[\partial_z u]}_\alpha+\overline{[\partial_x w]}+(\partial_x\varphi_\alpha-\psi_\alpha\partial_x z_\alpha)(z-z_\alpha)\\
&\qquad\qquad+\partial_x\psi_\alpha\left(\frac{(z-z_\alpha)^2}{2}-\frac{h_\alpha^2}{24}\right)\Bigg)^2\\
&+\left(\nu^0_{zz,\alpha}+\nu^1_{zz,\alpha}(z-z_\alpha)\right)\left(\overline{[\partial_z w]}_\alpha+\psi_\alpha(z-z_\alpha)\right)^2\Bigg]dz\\
&-\frac{1}{\rho}\left(\beta_0+\frac{\beta_1}{|\boldsymbol{U}|}\right)\left(1+(\partial_x b)^2\right)^{3/2}\left(u^+_{1/2}\right)^2\leq 0.
\end{aligned}
$$

Summary of models

Model	Disc. spaces $(u_\alpha, w_\alpha, q_\alpha)$	Dimension	Unknowns	Max. degree of derivatives	Approximated model
LIN-NH$_2$-STRESS	$(\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3)$	8L+1	H $\{\overline{u}_\alpha, \lambda_\alpha\}$ $\{\overline{w}_\alpha, \varphi_\alpha, \psi_\alpha\}$ $\{\overline{q}_\alpha, q_{\alpha-1/2}, \pi_\alpha\}$	2	Navier-Stokes
LIN-NH$_1$-STRESS	$(\mathbb{P}_1, \mathbb{P}_1, \mathbb{P}_2)$	6L+1	H $\{\overline{u}_\alpha, \lambda_\alpha\}$ $\{\overline{w}_\alpha, \varphi_\alpha\}$ $\{\overline{q}_\alpha, q_{\alpha-1/2}\}$	2	Navier-Stokes
LIN-H-STRESS	$(\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3)$	2L+1	H $\{\overline{u}_\alpha, \lambda_\alpha\}$	4	Navier-Stokes ($D_t w$ neglected)
LIN-NH$_2$	$(\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3)$	8L+1	H $\{\overline{u}_\alpha, \lambda_\alpha\}$ $\{\overline{w}_\alpha, \varphi_\alpha, \psi_\alpha\}$ $\{\overline{q}_\alpha, q_{\alpha-1/2}, \pi_\alpha\}$	1	Euler
LIN-NH$_1$	$(\mathbb{P}_1, \mathbb{P}_1, \mathbb{P}_2)$	6L+1	H $\{\overline{u}_\alpha, \lambda_\alpha\}$ $\{\overline{w}_\alpha, \varphi_\alpha\}$ $\{\overline{q}_\alpha, q_{\alpha-1/2}\}$	1	Euler

Table: Summary of models introduced in this work, discrete spaces for $u_\alpha(z), w_\alpha(z), q_\alpha(z)$, unknowns, maximum degree of the derivatives appearing in each model and the original model that is approximated.

In the case of granular flows, the viscosity is

$$\nu_{\text{an}}(z) = \frac{g \sin\theta \, (H - z)}{\sqrt{|\partial_z u|^2 /4 + \delta^2}},$$

being δ a regularisation parameter. The following profiles are obtained

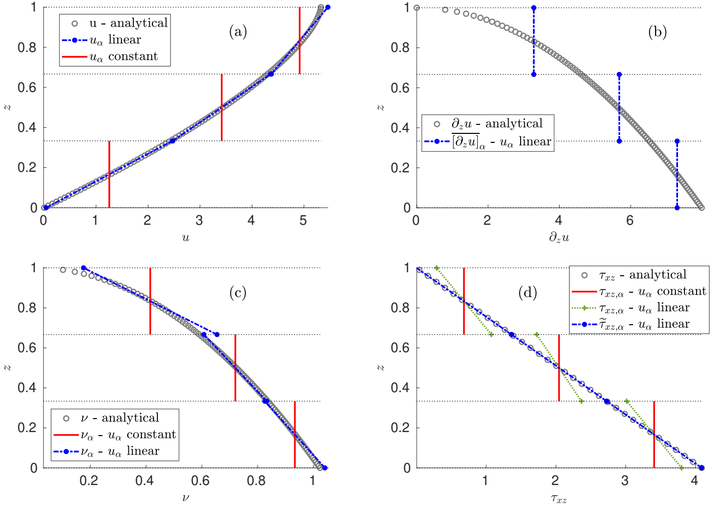
$$\left\{ \begin{aligned} u_{\text{an}}(z) &= \frac{2}{3} I_\theta \left(H^{3/2} - (H - z)^{3/2} \right), \\ \partial_z u_{\text{an}}(z) &= I_\theta \sqrt{H - z}, \\ \tau_{xz,\text{an}}(z) &= g \sin\theta \, (H - z), \end{aligned} \right. \tag{1}$$

with

$$I_\theta = \frac{I_0}{d_s} \left(\frac{\tan\theta - \mu_s}{\mu_2 - \tan\theta} \right) \sqrt{\varphi_s g \cos\theta},$$

being $d_s, \varphi_s, I_0, \mu_s, \mu_2$ constant parameters depending on the granular material.

Uniform flows: Dry granular flows with $\mu(I)$-rheology



Concerning Herschel-Bulkley fluids, the regularised viscosity coefficient is given by

$$\nu_{\rm an}(z) = \frac{\tau_y + K\left|\partial_z u\right|^n}{\sqrt{\left|\partial_z u\right|^2/4 + \delta^2}},$$

with τ_y, K and n constant rheological parameters.
The flow can be split into a lower sheared layer with height h_c defined by

$$h_c = H - \frac{\tau_y}{g\sin\theta},$$

and the (pseudo-)plug top layer, with thickness $H - h_c$. Then, the analytical solution reads for $z < h_c$

$$\left\{\begin{array}{l} u_{\rm an}(z) = u_{\rm plug}\left(1 - \left(1 - \dfrac{z}{h_c}\right)^{(n+1)/n}\right), \\[2ex] \partial_z u_{\rm an}(z) = \left(\dfrac{g\sin\theta}{K}\right)^{1/n}(h_c - z)^{1/n}, \\[2ex] \tau_{xz,{\rm an}}(z) = \rho g\sin\theta\,(H - z)\,, \end{array}\right. \tag{1}$$

and we have $u_{\rm an}(z) = u_{\rm plug}, \partial_z u_{\rm an} = 0$, and $\left|\tau_{xz,{\rm an}}(z)\right| \leq \tau_y$, for $z \geq h_c$ with

$$u_{\rm plug} = \frac{n}{(n+1)}\left(\frac{g\sin\theta}{K}\right)^{1/n} h_c^{(n+1)/n}.$$

In this case we consider a test where a material with height $H = 0.05$ and a slope with angle $\theta = 20^\circ$ are taken. The rheological parameters are $\tau_y = 0.033$, $K = 0.026$ and $n = 0.33$.

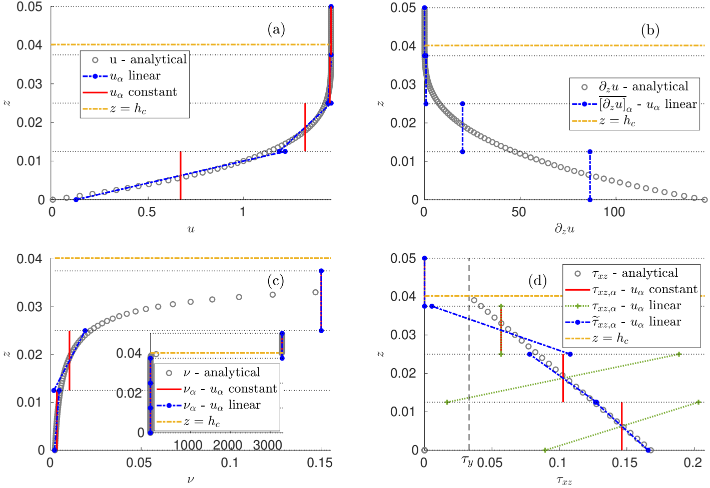


Figure: Comparison between the analytical vertical profiles (grey circles) and layerwise approximations with constant (solid red lines) and linear (dot-dashed blue and dotted green

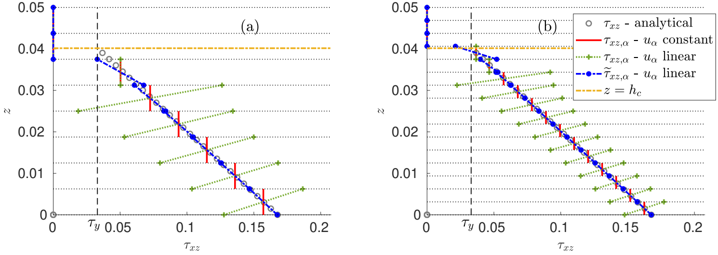


Figure: Comparison between the analytical vertical profiles (grey circles) and layerwise approximations with constant (solid red lines) and linear (dot-dashed blue and dotted green lines) horizontal velocity for the stress tensor component τ_{xz}, where $\widetilde{\tau}_{xz,\alpha}$ denotes the second-order correction of $\tau_{xz,\alpha}$. (a) 8 layers; (b) 16 layers.

Outline

- Introduction
- Laver-Averaged approximation of Navier-Stokes
- Dispersion relations

- **LDNH models** LDNH_0 $(\mathbb{P}_0, \mathbb{P}_0, \mathbb{P}_1)$ and LDNH_2 $(\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2)$

E.D. Fernández-Nieto, M. Parisot, Y. Penel, J. Sainte-Marie. A hierarchy of dispersive layer-averaged approximations of Euler equations for free surface flows. *Commun. Math. Sci.*, 16(05):1169–1202, 2018.

- **LIN-NH models** $\mathrm{LIN\text{-}NH}_1$ $(\mathbb{P}_1, \mathbb{P}_1, \mathbb{P}_1)$ and $\mathrm{LIN\text{-}NH}_2$ $(\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3)$

C. Escalante, E. Fernandez-Nieto, J. Garres-Díaz, T. Morales de Luna, Y. Penel. Non-hydrostatic layer-averaged approximation of Euler system with enhanced dispersion properties. *Computational and Applied Mathematics*, 177,42, 2023.

Dispersion relation of LDNH models

Lemma

There exists a plane wave solution $\left(\hat{H}, \hat{u}_\alpha, \hat{w}_\alpha, \hat{q}_\alpha\right) e^{i(kx-\omega t)}$ to the linearised version of $(\mathrm{LDNH}_\nu)_{\nu\in\{0,2\}}$ provided the following dispersion relation holds

$$\omega^2 = k^2 c_{sw}^2 \left\langle \mathcal{A}_{kH_0}^{-1}\boldsymbol{e}, \boldsymbol{\ell} \right\rangle, \tag{1}$$

where $c_{sw} = \sqrt{gH_0}$s, $\boldsymbol{e} = (1,\ldots,1) \in \mathbb{R}^N$, $\mathcal{A}_x = \mathcal{I}_L + x^2\mathcal{B}(\{l_\alpha\}_\alpha)$.

Theorem

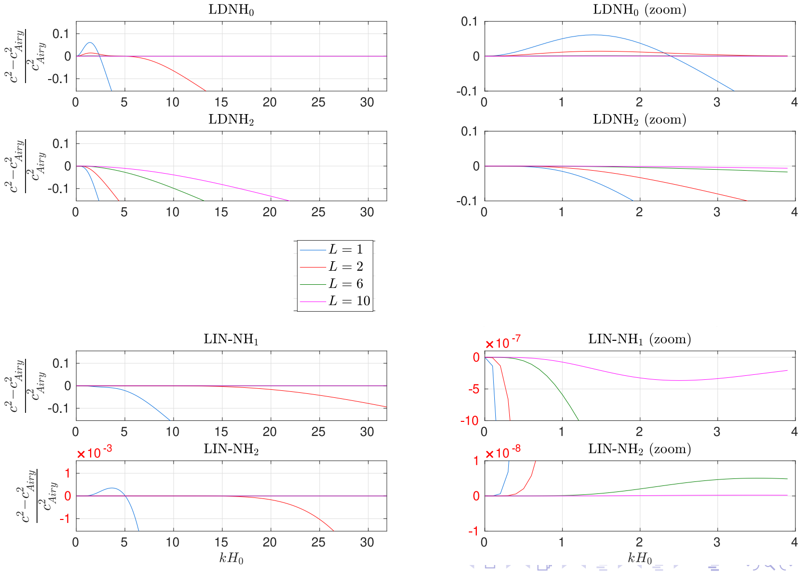
- For a given number of layers $N \geq 4$, the dispersion relation (1) can be made explicit in the homogeneous case $\ell_\alpha = \frac{1}{N}$,

$$c_N^2(kH_0) = \frac{\omega^2}{k^2 gH_0} = \frac{\mathcal{P}_N(kH_0)}{\mathcal{Q}_N(kH_0)}.$$

- When the number N of layers increases, the celerity c_N converges to the celerity associated to the Euler equations obtained from the Airy wave theory:

$$c_{Airy}^2(kH_0) = \frac{\tanh(kH_0)}{kH_0}.$$

Comparison of dispersion relations



Conclusions

- Several layer-averaged models with layerwise linear horizontal velocity and non-hydrostatic pressure for the Navier-Stokes system are proposed.
- We focus on an appropriate definition of the terms that come from viscous contributions for a general stress tensor.
- In particular, we give detailed definitions of all components of the stress tensor when it is proportional to the strain rate tensor.
    - In that case, the approximations of the derivatives of the velocity are inspired by the theory of distributions, in order to account for the possible discontinuities of the velocity at the internal interfaces.
- These models satisfy a dissipative energy balance, where the right-hand side is written in integral form.

- An important remark is the fact that all terms in these models are approximated to second-order accuracy, except for $\tau_{xz,\alpha}$. Concretely, it is a first-order approximation, due to the fact that $\partial_z u$ is layerwise constant. However, we also propose a correction allowing us to obtain the second-order accuracy. This corrected model satisfies a dissipative energy balance up to second order.
- These and other models are also obtained from an asymptotic analysis of the Navier-Stokes system, for different orders of magnitude of the shallowness parameter (ε).
- The proposed layerwise linear approach is effective for some geophysical flows, including complex viscoplastic fluids, in the uniform configuration, where it is possible to get analytical solutions.
- Previous models with piecewise constant horizontal velocity are particular cases.
- Enhanced dispersion relation for LDNH and LIN-NH models.

Modelling and dispersion relations of layer-averaged non-hydrostatic Euler equations

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